Continuity of Bowen-Margulis Currents for Hyperbolic Groups

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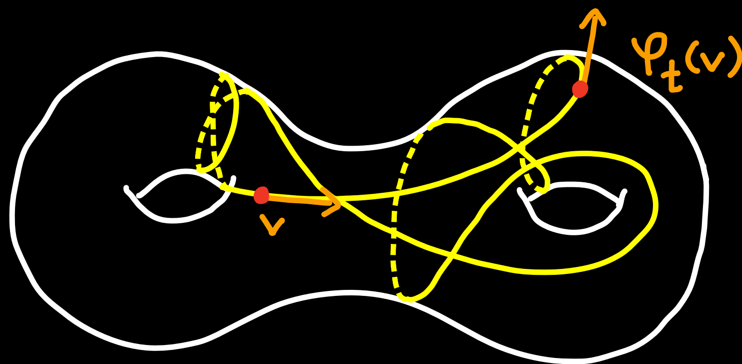
MOTIVATION

(M,g): closed, negatively curved manifold $\quad \Gamma = \pi_1(M)$

- **GEODESIC FLOW:** $\varphi_t = \varphi_t^g : T^1M \circlearrowleft$
  Anosov!!!



- **BOWEN-MARGULIS MEASURE:**
  $m = m_g$: unique (φ_t^g)-invariant
  prob. measure maximizing the entropy

THM (KATOK-KNIEPER-POLLICOTT-WEISS '89)

B-M map $\quad g \longmapsto m_g$ is continuous

$g \in$ {neg. curved metrics} + C^2 top

$m_g \in$ { prob. measures on T^1S } + weak* top

GOAL: Similar result for more general g, and M, ... and Γ!

Example: Surface case

S closed surface of $\chi(S)<0$

$$\mathcal{T}_{<0}(S) = \left\{ \begin{array}{c} \text{marked, neg. curved Riem.} \\ \text{metrics on } S \end{array} \right\} \Big/ \text{isotopy}$$

$$= \left\{ \begin{array}{l} \text{neg. curved Riem. metrics } \tilde{g} \text{ on } \tilde{S} \\ + \text{ isometric action } \Gamma \overset{\pi}{\curvearrowright} (\tilde{S},\tilde{g}) = \tilde{S}_g \\ \quad \text{by Deck transf.} \end{array} \right\} \Big/ \sim$$

$(\pi,\tilde{g}) \sim (\pi',\tilde{g}') \longleftrightarrow \exists\, \lambda>0$ and Γ-equiv. homeo $\tilde{S}_g \xrightarrow{F} \tilde{S}_{g'}$

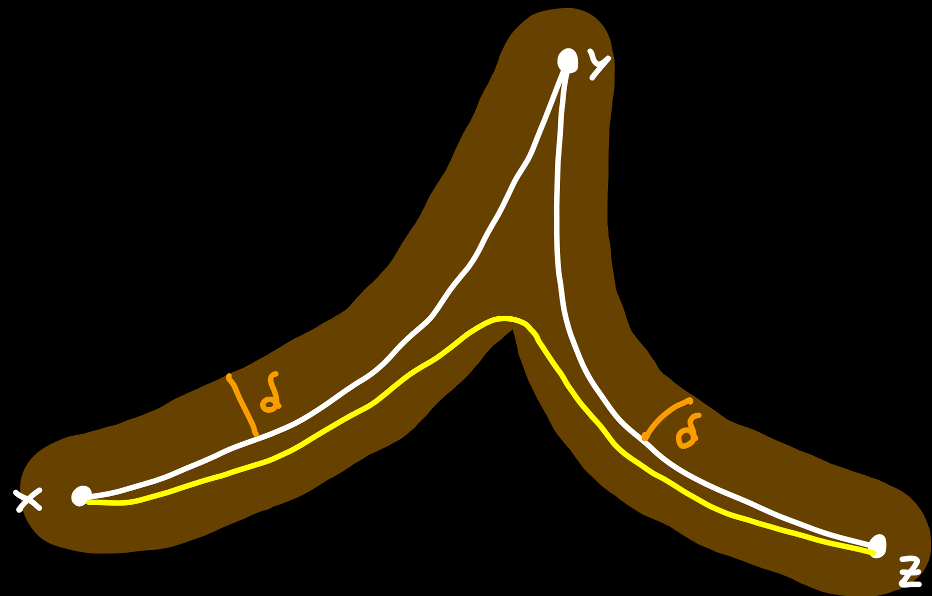
$$d_{\tilde{g}'}(Fx,Fy) = \lambda\, d_{\tilde{g}}(x,y) \quad \forall\, x,y$$

- **Thurston's Lipschitz Metric:**

$$d_{Th}((\pi,\tilde{g}),(\pi',\tilde{g}')) := \operatorname{Log}\inf \left\{ L \geq 0 \,\middle|\, \begin{array}{l} \exists\, \tilde{S}_g \xrightarrow{F} \tilde{S}_{g'} \ \Gamma\text{-equiv. homeo} \\ d_{\tilde{g}}(x,y) \leq L\, d_{\tilde{g}'}(Fx,Fy) \ \forall x,y \end{array} \right\}$$

$$\boxed{\Delta([\pi,\tilde{g}],[\pi',\tilde{g}']) := d_{Th}((\pi,\tilde{g}),(\pi',\tilde{g}')) + d_{Th}((\pi',\tilde{g}'),(\pi,\tilde{g})) \quad \text{metric on } \mathcal{T}_{<0}(S)}$$

HYPERBOLIC SPACES & GROUPS



- **GROMOV HYPERBOLICITY:**

  X geodesic metric space is

  δ-hyperbolic ($\delta \geq 0$) if $\forall x, y, z \in X$

  $[x,z] \subset N_\delta([x,y] \cup [y,z])$

- **HYPERBOLIC GROUPS:**

  Γ finitely generated group acting

  properly & coboundedly by isometries $\}$ geometric action

  on some Gromov hyperbolic space X

EXAMPLES

- Finite groups ($X = \{pt\}$), $\mathbb{Z}$ ($X = \mathbb{R}$)
- Free groups: $\Gamma = \pi_1(= R_n)$ ($X = \widetilde{R}_n = \text{tree}$)
- $\Gamma = \pi_1(M)$, M_g closed neg. curved mfd ($X = \widetilde{M}_g$)
- $\Gamma = \mathbb{Z}^2$ is not hyperbolic ($\mathbb{R}^n$ is not Gromov hyperbolic)

The Space of Metric Structures

Γ hyperbolic group, non-elementary

$$\mathcal{D}_\Gamma := \left\{ \Gamma \overset{\pi_X}{\curvearrowright} X \;\middle|\; \begin{array}{l} \text{geometric action on the} \\ \text{Gromov hyperbolic space } X \end{array} \right\} \Big/ \sim$$

$$X \sim Y \longleftrightarrow \exists\, \lambda > 0,\ A \geq 0, \text{ and } \Gamma\text{-equiv. map } X \xrightarrow{F} Y$$
$$|d_Y(Fx, Fy) - \lambda d_X(x,y)| \leq A \quad \forall\, x, y \in X$$

- **Metric on $\mathcal{D}_\Gamma$:**

$$\mathrm{Dil}(X,Y) := \operatorname{Log} \inf \left\{ L \geq 0 \;\middle|\; \begin{array}{l} \exists\ X \xrightarrow{F} Y \ \Gamma\text{-equiv. map},\ A \geq 0 \\ \text{s.t.} \quad d_X(x,y) \leq L\, d_Y(Fx, Fy) + A \quad \forall x, y \in X \end{array} \right\}$$

$$\boxed{\Delta([X],[Y]) := \mathrm{Dil}(X,Y) + \mathrm{Dil}(Y,X)}$$

EXAMPLES & PROPERTIES

- S closed hyperbolic surface $\Rightarrow \mathcal{T}_{<0}(S) \hookrightarrow \mathcal{D}_{\pi_1(S)}$

- Γ = free group $\Rightarrow \left\{ \begin{array}{c} \text{minimal, geometric actions} \\ \text{of } \Gamma \text{ on metric trees} \end{array} \right\} \Big/ \begin{array}{c} \Gamma\text{-equiv.} \\ \text{homothety} \end{array} \hookrightarrow \mathcal{D}_\Gamma$

  $\shortparallel$ Culler-Vogtmann Outer space

- Metric structures in $\mathcal{D}_\Gamma$ can be induced by

  1) Nice random walks on Γ (BLACHÈRE-HAÏSSINSKY-MATHIEU '11)

  2) Anosov representations of Γ (DEY-KAPOVICH '19)

- $(\mathcal{D}_\Gamma, \Delta)$ is contractible, unbounded, separable (O.R. '22)

- $(\mathcal{D}_\Gamma, \Delta)$ is geodesic (CANTRELL-O.R. '22)

Quasiconformal Measures

• **Gromov Boundary:** X Gromov hyperbolic, $o \in X$

$\rightsquigarrow \partial X = \{\text{geodesic rays based at } o\}/\text{finite Hausdorff distance}$

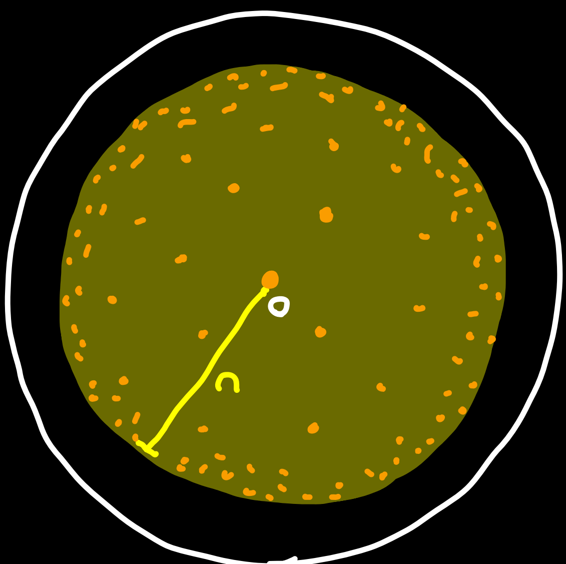
Γ hyperbolic group $\rightsquigarrow \partial\Gamma := \partial X$, $[X] \in \mathcal{D}_\Gamma$

$\rightsquigarrow$ well defined, compact metrizable

+ topological action $\Gamma \curvearrowright \partial\Gamma$

• **Quasiconformal Measures:** $[X] \in \mathcal{D}_\Gamma$, $o \in X$

$$\nu_X = \text{“}\lim_{n\to\infty}\text{”} \frac{1}{\{g\in\Gamma \mid d_X(go,o)\le n\}} \sum_{\substack{g\in\Gamma \\ d_X(go,o)\le n}} \delta_{go} \in \mathrm{Prob}(\partial\Gamma)$$



$\rightsquigarrow$ Γ-quasi-invariant, ergodic

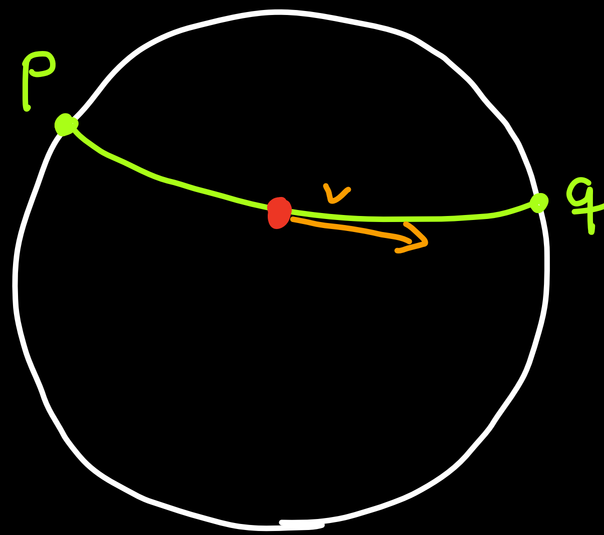
$\rightsquigarrow$ Measure class of ν_X only depends on $[X] \in \mathcal{D}_\Gamma$

Fact: $\nu_X \sim \nu_Y$ iff $[X] = [Y]$

BOWEN-MARGULIS CURRENTS

$\partial^2\Gamma = \{(p,q) \in \partial\Gamma \times \partial\Gamma \mid p \neq q\} \circlearrowleft \Gamma$ diagonal action

" $\partial^2\Gamma \times \mathbb{R} \rightsquigarrow T^1\widetilde{S}_g$ "



- **GEODESIC CURRENTS:**

$\mathcal{C}urr(\Gamma) = \{$ Radon, Γ-invariant measures on $\partial^2\Gamma\}$ + weak* top

$\mathbb{P}\mathcal{C}urr(\Gamma) = (\mathcal{C}urr(\Gamma) - \{0\})/\mathbb{R}^+$ + quotient top (BONAHON '91: compact metrizable)

THM (FURMAN '02):

1) $[x] \in \mathfrak{D}_\Gamma \Rightarrow \exists!\ BM[x] = [m] \in \mathbb{P}\mathcal{C}urr(\Gamma)$ s.t. $m \sim \nu_x \otimes \nu_x$

2) $BM : \mathfrak{D}_\Gamma \longrightarrow \mathbb{P}\mathcal{C}urr(\Gamma)$ is injective

THM (CANTRELL-TANAKA '21): ν_x "maximizes" the "entropy"

MAIN RESULT

THM (O.R. '22): $\forall \delta \geq 0$, Bowen-Margulis map

$$BM: \mathfrak{D}_\Gamma^\delta \longrightarrow \mathbb{P}\mathcal{C}urr(\Gamma) \text{ is continuous}$$

$$\overset{\shortparallel}{\{[X] \mid X \text{ is } \delta\text{-hyperbolic } \& \ v_X = 1\}}$$

exponential growth rate of $\Gamma \curvearrowright X$

EXAMPLES

- S surface $\Longrightarrow$ Teichmüller space $= \mathcal{T}_{-1}(S) \subset \mathfrak{D}_{\pi_1(S)}^{\log 2}$
- S surface $\Longrightarrow$ Quasi-Fuchsian space $\mathcal{QF}(S) \subset \mathfrak{D}_{\pi_1(S)}^{\log 4}$
- Bounded subsets of $\mathcal{T}_{<0}(M)$ contained in $\mathfrak{D}_{\pi_1(M)}^{\delta}$ for some δ

  (recovers KKPW)
- Γ free group then $\mathcal{CV}(\Gamma) = \mathfrak{D}_\Gamma^0$

  (recovers KAPOVICH-NAGNIBEDA '07)

Sketch of Proof

Suppose $\mathcal{D}^{\delta}_{\Gamma} \supset [X_n]_n \xrightarrow{n\to\infty} [X_\infty]$

1) Bochi-Type inequality (Breuillard '18 - Fujiwara) $\Longrightarrow$ $\exists\, D \geq 0$ s.t. $\forall n$ $\exists$ $Y_n \in [X_n]$ s.t.

i) Y_n δ-hyp. $v_{Y_n} = 1$, codiameter $\leq D$

ii) $\exists$ $Y_1 \xrightarrow{F_n} Y_n$ Γ-equiv. s.t.

$$\Lambda_n^{-1} d_{Y_1}(x,y) - D \leq d_{Y_n}(F_n x, F_n y) \leq \Lambda_n d_{Y_1}(x,y) + D$$

$\Longrightarrow$ $Y_\infty =$ "$\mathrm{Lim}(F_n(Y_1), d_{Y_n})$" $\in [X_\infty]$

2) Up to subsequence, i) $\nu_{Y_n} \xrightarrow{*} \nu_{Y_\infty}$

ii) $\forall g \in \Gamma$ $\quad \dfrac{dg\nu_{Y_n}}{d\nu_{Y_n}}$ "$\longrightarrow$" $\dfrac{dg\nu_{Y_\infty}}{d\nu_{Y_\infty}}$

3) Up to subsequence and rescaling,

i) $\exists$ $dm_n = G_n\, d\nu_{Y_n} d\nu_{Y_n} \in BM[X_n]$ s.t.
G_n "$\longrightarrow$" G_∞ for some G.

ii) $dm_\infty := G_\infty\, d\nu_{Y_\infty} d\nu_{Y_\infty} \in BM[X_\infty]$

iii) $m_n \xrightarrow{*} m_\infty$

$\square$

Thank You!!!