Matrix Products Inequalities, the Berger-Wang Identity, and Non-Positive Curvature

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MOTIVATION

BOCHI-TYPE INEQUALITY (O.-R.'16): $A, B \in M_2(\mathbb{R})$

$$\|AB\| \leq 8\|A\|\|B\| \cdot \underbrace{\max\left\{\frac{\rho(A)}{\|A\|}, \frac{\rho(B)}{\|B\|}, \frac{\rho(AB)}{\|A\|\|B\|}\right\}^{1/2}}_{\circledast}$$

$\circledast$ non-trivial term

ρ = spectral radius

$\|\cdot\|$ = operator norm

COROLLARY: A, B, AB nilpotent $\Longrightarrow AB = 0$

Proof: Hyperbolic geometry

QUESTION: What about higher dimensions??

Theorem 1 (O.-R. '17): $\forall d \geq 1, \exists C, N, r \geq 1$ st $\forall A_1, \ldots, A_N \in M_d(\mathbb{R})$

$$\|A_1 \cdots A_N\| \leq C \|A_1\| \cdots \|A_N\| \underbrace{\left(\max_{1 \leq \alpha < \beta \leq N} \frac{\rho(A_\alpha A_{\alpha+1} \cdots A_\beta)}{\|A_\alpha\| \|A_{\alpha+1}\| \cdots \|A_\beta\|} \right)^{1/r}}_{\circledast \text{ non-trivial term}}$$

"If $\|A_1 \cdots A_N\| \approx \|A_1\| \cdots \|A_N\|$, then $\|A_\alpha\| \cdots \|A_\beta\| \approx \rho(A_\alpha \cdots A_\beta)$ for some $1 \leq \alpha \leq \beta \leq N$"

Corollary: If $\circledast = 0$

$$\left(\begin{array}{c} \text{each subproduct} \\ A_\alpha A_{\alpha+1} \cdots A_\beta \\ \text{is nilpotent} \end{array} \right) \Longrightarrow A_1 \cdots A_N = 0$$

SMALL CASE: $d=3 \Rightarrow N=5,\ r=2^{188}$

$$\|ABCDE\| \leq C_3 \|A\|\|B\|\|C\|\|D\|\|E\| \max \left\{ \begin{array}{c} \frac{\rho(A)}{\|A\|}, \frac{\rho(B)}{\|B\|}, \frac{\rho(C)}{\|C\|}, \frac{\rho(D)}{\|D\|}, \frac{\rho(E)}{\|E\|}, \\ \frac{\rho(AB)}{\|A\|\|B\|}, \frac{\rho(BC)}{\|B\|\|C\|}, \frac{\rho(CD)}{\|C\|\|D\|}, \frac{\rho(DE)}{\|D\|\|E\|}, \\ \frac{\rho(ABC)}{\|A\|\|B\|\|C\|}, \frac{\rho(BCD)}{\|B\|\|C\|\|D\|}, \frac{\rho(CDE)}{\|C\|\|D\|\|E\|}, \\ \frac{\rho(ABCD)}{\|A\|\|B\|\|C\|\|D\|}, \frac{\rho(BCDE)}{\|B\|\|C\|\|D\|\|E\|}, \\ \frac{\rho(ABCDE)}{\|A\|\|B\|\|C\|\|D\|\|E\|} \end{array} \right\}^{1/2^{188}}$$

RMK: In general, $N \leq \prod_{i=1}^{d} \binom{d}{i}$, $r \leq (Nd+1)^{Nd^2+2}$

C_d computable...

Application: Berger-Wang Identity

$S \subset M_d(\mathbb{R})$ bounded $\quad \|S\| := \sup\{\|A\| : A \in S\}$

$\rho(S) := \sup\{\rho(A) : A \in S\}$

$S^n := \{A_1 A_2 \cdots A_n \mid A_i \in S\}$

Joint Spectral Radius

$$\mathcal{R}(S) := \lim_{n\to\infty} \|S^n\|^{1/n}$$

Thm 1 implies:

$$\boxed{\|S^N\| \leqslant C \|S\|^{N-1/r} \max_{1\leqslant j\leqslant N} \rho(S^j)^{1/jr}}$$

Bochi type inequality

Corollary (Berger-Wang '92):

$$\boxed{\mathcal{R}(S) := \overline{\lim_{n\to\infty}}\, \rho(S^n)^{1/n}}$$

(Apply to S^n + let $n \to \infty$)

Markovian Berger-Wang (Dai '13, Kozyakin '14): $S=\{A_1,\dots,A_N\}$ $\Sigma\in M_N(\{0,1\})$

$$S_\Sigma^n := \{A_{i_1}\cdots A_{i_n} \mid \Sigma_{i_j i_{(j+1)}}=1,\ \Sigma_{i_n \ell}=1 \text{ for some } \ell\}$$

$$\mathcal{R}_\Sigma(S) := \lim_{n\to\infty} \|S_\Sigma^n\|^{1/n} \overset{BW}{=} \overline{\lim_{n\to\infty}}\, \rho(S_\Sigma^n)^{1/n}$$

Related Work

- Bochi '03 ⇝ Bochi inequalities
- Morris '11, '13, '16 ⇝ Ergodic Theory of B-W + Bochi inequalities for singular value pressure

- O.-R. '16
- Breuillard-Fujiwara '18

} Bochi inequalities for isometries of non-positively curved spaces

- Breuillard-Sert '18 ⇝ Joint spectrum (all singular values)
- Breuillard '21 ⇝ Optimal bounds for Bochi inequalities

Main Result

Theorem 2 (O.-R. '21): $\forall d \geqslant 1, \forall \ell \geqslant 1, \exists \tilde{C}, \tilde{N}, \tilde{r} \geqslant 1$ st $\forall A_1, \ldots, A_{\tilde{N}} \in M_d(\mathbb{R}), \exists\, 0 \leqslant n_0 < n_1 < \cdots < n_\ell \leqslant \tilde{N}$ st

$$\|A_1 \cdots A_{\tilde{N}}\| \leqslant \tilde{C} \prod_{i=1}^{\tilde{N}} \|A_i\| \left(\prod_{0 \leqslant \alpha < \beta \leqslant \ell} \frac{\rho(A_{n_\alpha+1} \cdots A_{n_\beta})}{\prod_{n_\alpha < t \leqslant n_\beta} \|A_t\|} \right)^{1/\tilde{r}}$$

"If $\|A_1 \cdots A_{\tilde{N}}\| \approx \|A_1\| \cdots \|A_N\|$, then $\|A_{n_\alpha+1}\| \cdots \|A_{n_\beta}\| \approx \rho(A_{n_\alpha+1} \cdots A_{n_\beta})$ for <u>all</u> $0 \leqslant \alpha < \beta \leqslant \ell$"

Can deal with multiple singular values simultaneously!

Application 1: Symmetric Spaces

$$A \in SL_d(\mathbb{R}) \Rightarrow \boxed{\begin{aligned} \kappa(A) &:= \sqrt{\sum_{i=1}^{d} [\operatorname{Log} \underbrace{\sigma_i(A)}_{\text{singular values}}]^2} \\ \chi(A) &:= \sqrt{\sum_{i=1}^{d} [\operatorname{Log}|\underbrace{\chi_i(A)}_{\text{eigenvalues}}|]^2} \end{aligned}}$$

metric on $SL_d(\mathbb{R})/SO_d(\mathbb{R})$

($d=2$ gives hyperbolic plane)

$$S \subset SL_d(\mathbb{R}) \Rightarrow \mathcal{K}(S) := \lim_{n\to\infty}\left(\sup_{A_1,\dots,A_n\in S} \frac{\kappa(A_1\cdots A_n)}{n}\right)$$

Berger-Wang for Symmetric Spaces

(Breuillard-Fujiwara '18): $S \subset SL_d(\mathbb{R})$ bounded, then

$$\mathcal{K}(S) = \overline{\lim_{n\to\infty}}\left(\sup_{A_1,\dots,A_n\in S} \frac{\chi(A_1\cdots A_n)}{n}\right)$$

Markovian version also holds!!!

Application 2: Singular Value Pressure

$A \in M_d(\mathbb{R})$
$s > 0$

$$\boxed{\begin{aligned} \varphi^s(A) &:= \begin{cases} \sigma_1(A) \cdots \sigma_q(A)\, \sigma_{q+1}(A)^{s-q}, & q < s \le q+1 \le d \\ |\det(A)|^{s/d} & s \ge d \end{cases} \\ \gamma^s(A) &:= \begin{cases} |\chi_1(A) \cdots \chi_q(A)\, \chi_{q+1}(A)^{s-q}|, & q < s \le q+1 \le d \\ |\det(A)|^{s/d} & s \ge d \end{cases} \end{aligned}}$$

*Singular Value Pressure: $s > 0$, $A_1, \dots, A_k \in M_d(\mathbb{R})$

$$P((A_1, \dots, A_k), s) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{i_1, \dots, i_n = 1}^{k} \varphi^s(A_{i_1} \cdots A_{i_n}) \right)$$

Berger-Wang for S.V.P.

$\forall s > 0$, $A_1, \dots, A_k \in M_d(\mathbb{R})$

$$P((A_1, \dots, A_k), s) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log \left(\sum_{i_1, \dots, i_n = 1}^{k} \gamma^s(A_{i_1} \cdots A_{i_n}) \right)$$

Thank You!!!