The Space of Metric Structures on Hyperbolic Groups

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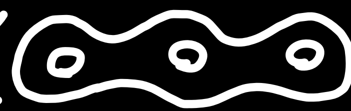
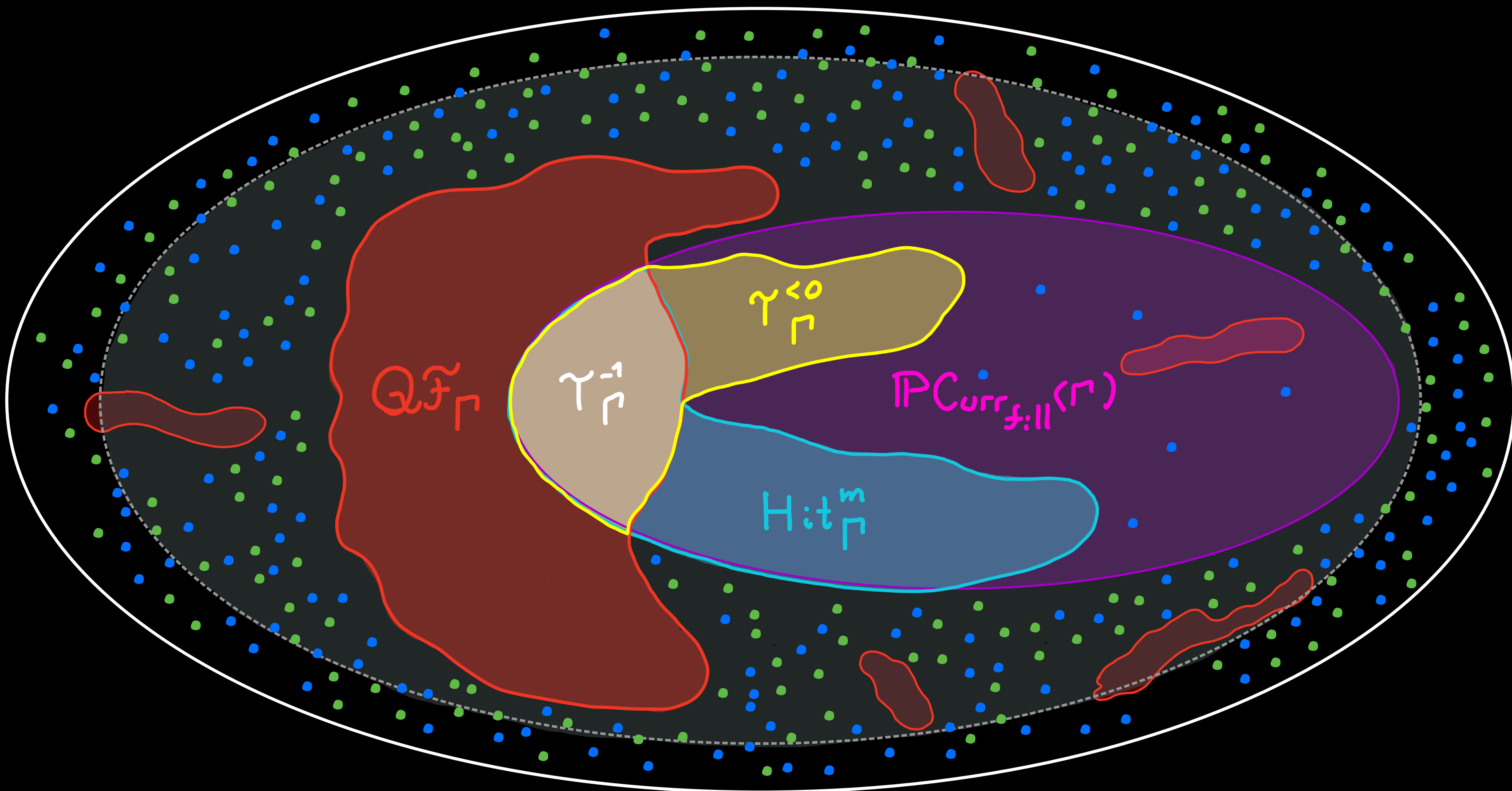
EXAMPLE: $\mathfrak{D}_\Gamma$ for $\Gamma = \pi_1($$)$



$\mathfrak{D}_\Gamma$ PARAMETRIZES GEOMETRIC ACTIONS OF Γ ON HYPERBOLIC SPACES

Motivation: Deformation Spaces

Γ finitely generated group

- **Teichmüller spaces** — $\mathcal{T}^{-1}_{\Gamma}$ — $\Gamma \curvearrowright \mathbb{H}^2$

  Ahlfors, Bers, Bonahon, Fricke, Gardiner, Masur, McMullen, Thurston...

  $\mathcal{T}^{<0}_{\Gamma}$ — $\Gamma \curvearrowright$ contractible neg. curved Riemannian manifolds

- **Outer spaces** — $\mathcal{CV}_{\Gamma}$ — $\Gamma \curvearrowright$ trees

  Culler, Guirardel, Horbez, Levitt, Morgan, Paulin, Shalen, Vogtmann...

  ($\mathbb{R}$-trees, CAT(0) cube complexes)

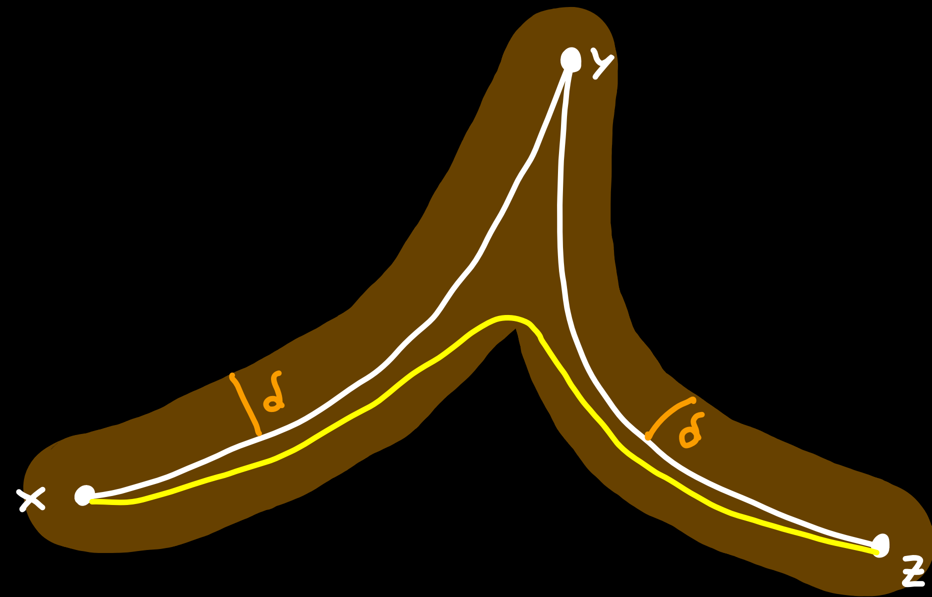
- **Character varieties** — $\mathcal{QF}_{\Gamma}$, $\mathrm{Hit}^{d}_{\Gamma}$ — $\Gamma \curvearrowright$ symmetric spaces

  convex-cocompact/Anosov

  Bridgeman, Burger, Guichard, Hitchin, Iozzi, Labourie, Wienhard...

Hyperbolic Spaces & Groups

- **Gromov Hyperbolicity:**
  X geodesic metric space is
  δ-hyperbolic ($\delta \geq 0$) if $\forall x, y, z \in X$
  $$[x,z] \subset N_\delta([x,y] \cup [y,z])$$



- **Hyperbolic Groups:**
  Γ finitely generated group acting
  properly & coboundedly by isometries $\}$ geometric action
  on some Gromov hyperbolic space X

Examples

- Finite groups ($X = \{pt\}$), $\mathbb{Z}$ ($X = \mathbb{R}$)
- Free groups: $\Gamma = \pi_1($  $= R_n)$ ($X = \widetilde{R}_n = \text{tree}$)
- $\Gamma = \pi_1(M)$, M_g closed neg. curved mfd ($X = \widetilde{M}_g$)
- Small cancellation groups

The Space of Metric Structures (Furman 2002)

Γ hyperbolic group, non-elementary

$$\mathcal{D}_\Gamma := \left\{ \Gamma \overset{\pi_X}{\curvearrowright} X \;\middle|\; \begin{array}{l}\text{geometric action on the} \\ \text{Gromov hyperbolic space } X\end{array} \right\} \Big/ \sim$$

$$X \sim Y \longleftrightarrow \exists\ \lambda > 0,\ A \geq 0, \text{ and } \Gamma\text{-equiv. map } X \xrightarrow{F} Y$$
$$|d_Y(Fx, Fy) - \lambda d_X(x,y)| \leq A \quad \forall\ x,y \in X$$

- **Metric on $\mathcal{D}_\Gamma$:**

$$\mathrm{DiL}(X,Y) := \operatorname{Log} \inf \left\{ L \geq 0 \;\middle|\; \begin{array}{l} \exists\ X \xrightarrow{F} Y \ \Gamma\text{-equiv. map},\ A \geq 0 \\ \text{s.t.}\quad d_X(x,y) \leq L\, d_Y(Fx, Fy) + A \ \ \forall x,y \in X \end{array} \right\}$$

$$\boxed{\Delta([X],[Y]) := \mathrm{DiL}(X,Y) + \mathrm{DiL}(Y,X)}$$

Examples of Metric Structures

- Γ surface group $\Longrightarrow$ $\mathcal{T}^{-1}_\Gamma \hookrightarrow \mathcal{T}^{<0}_\Gamma \hookrightarrow \mathcal{D}_\Gamma$ (Otal '90, Croke '90)
- Γ free group $\Longrightarrow$ $\mathcal{CV}_\Gamma \hookrightarrow \mathcal{D}_\Gamma$ (Francaviglia-Martino '11)
- Cayley graphs: $S \subset \Gamma$ finite generating set $\Longrightarrow [\mathrm{Cay}(\Gamma, S)] \in \mathcal{D}_\Gamma$
- Nice random walks on Γ (Blachère-Haïssinsky-Mathieu '11)

  $(Z_n)_n$ admissible random walk $\Longrightarrow$ $d_G(g,h) = -\mathrm{Log}\, \mathbb{P}(\exists n \mid g Z_n = h)$ (Green metric)

- Anosov representations of Γ (Dey-M.Kapovich '19, Cantrell-Tanaka '22)

  $\Gamma \overset{\rho}{\to} PSL_m(\mathbb{R})$ 1-dominated $\Longrightarrow$ $d_\rho(g,h) = \mathrm{Log}(\|\rho(g^{-1}h)\| \|\rho(h^{-1}g)\|)$

Example

Γ surface group

$$\begin{tikzcd} & \mathcal{T}^{<0}_\Gamma \arrow[dr] & & \\ \mathcal{T}^{-1}_\Gamma \arrow[ur] \arrow[dr] & & \mathbb{P}\mathcal{C}urr_{fill}(\Gamma) \arrow[r, hook] & \mathcal{D}_\Gamma \\ & Hit^m_\Gamma \arrow[ur] & & \end{tikzcd}$$

projective filling currents

Bridgeman-Canary-Labourie-Sambarino '15

Sapir '22, De Rosa-Martínez Granado '22

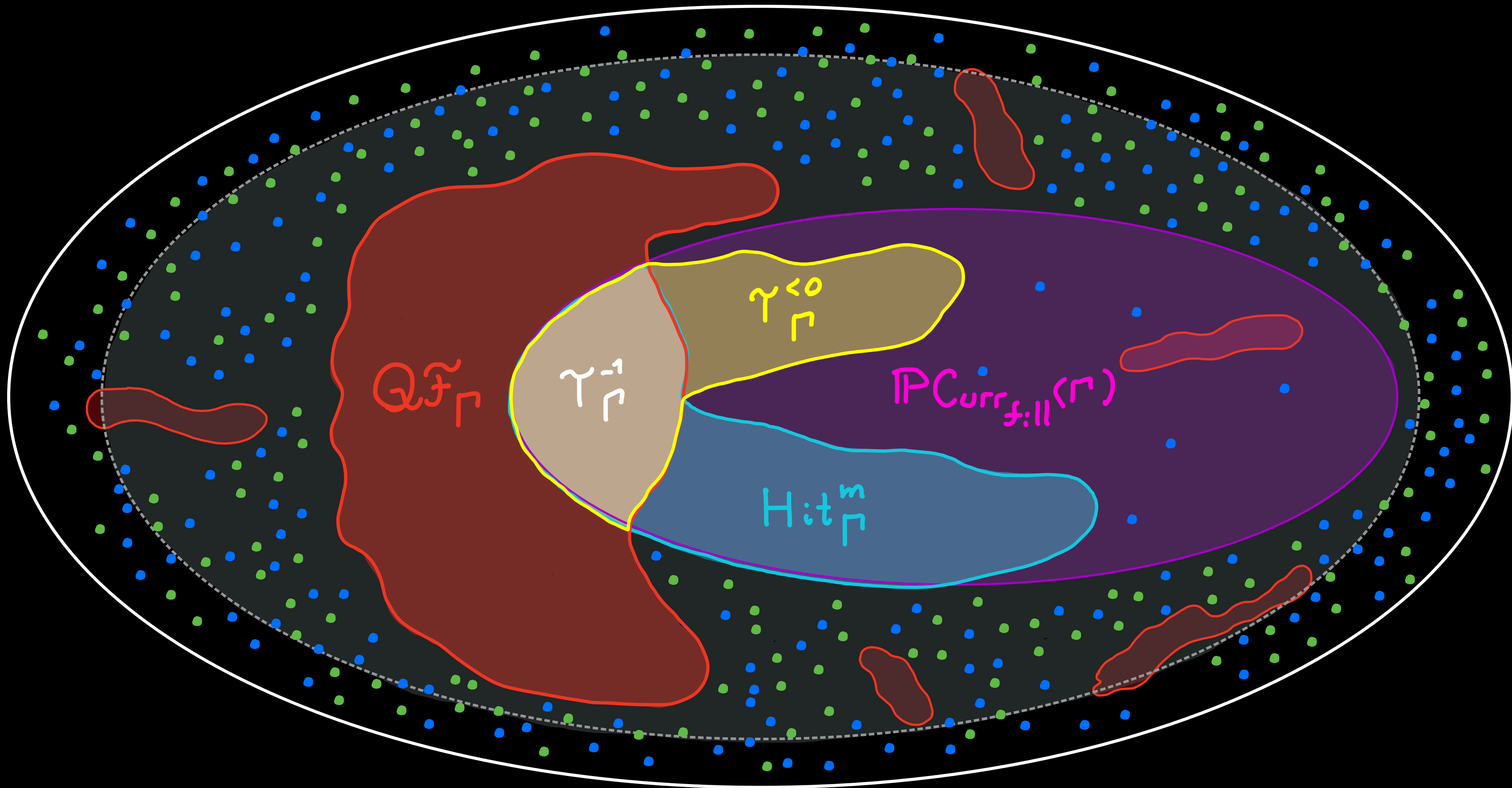
Properties

(R.'22, Cantrell-R.'22, Cantrell-R.'23)

- $(\mathfrak{D}_\Gamma, \Delta)$ is contractible, unbounded, separable
- $(\mathfrak{D}_\Gamma, \Delta)$ is geodesic
- $(\mathfrak{D}_\Gamma, \Delta)$ is not locally compact
- Natural action of $\mathrm{Out}(\Gamma)$ on $\mathfrak{D}_\Gamma$ is metrically proper
- $\mathfrak{D}_\Gamma$ has a "thick part" $\mathfrak{D}_\Gamma^{\delta, D} = \left\{ [X] \;\middle|\; \begin{array}{l} X \ \delta\text{-hyperbolic} \\ + \ \Gamma \curvearrowright X \text{ of codiameter} \leq D \\ \& \text{ critical exponent } 1 \end{array} \right\}$

$$\Gamma \text{ torsion-free} \implies \mathrm{Out}(\Gamma) \curvearrowright \mathfrak{D}_\Gamma^{\delta, D} \text{ cocompact}$$

EXAMPLE: $\mathfrak{D}_\Gamma$ for $\Gamma = \pi_1(\ )$



- Cayley graphs (dense)
- Green metrics (dense, Cantrell-R. '23)
- $\mathbb{P}\mathrm{Curr}_{fill}(\Gamma)$ (= $\overline{\text{cubulations with cyclic wall stabilizers}}$)
- $\overline{\text{cubulations}}$ (contains $\mathcal{QF}_\Gamma$, Brody-R. '23)
- Anosov reps

THEOREM (CANTRELL-REYES '22)

$\rho \neq \rho_* \in \mathfrak{D}_\Gamma \Longrightarrow \exists$ bi-infinite geodesic $\rho : \mathbb{R} \to \mathfrak{D}_\Gamma$

s.t. $\rho(0) = \rho$, $\rho(\Delta(\rho,\rho_*)) = \rho_*$

SKETCH OF PROOF: $\rho = [X], \rho_* = [X_*]$ $o \in X$, $o_* \in X_*$

$\hookrightarrow$ Define $d(g,h) = d_X(go, ho)$, $d_*(g,h) = d_{X_*}(go_*, ho_*)$ hyperbolic pseudometrics on Γ

$\hookrightarrow$ $t \in [0,1] \rightsquigarrow$ $\boxed{d_t := t\, d_* + (1-t)\, d}$ hyperbolic

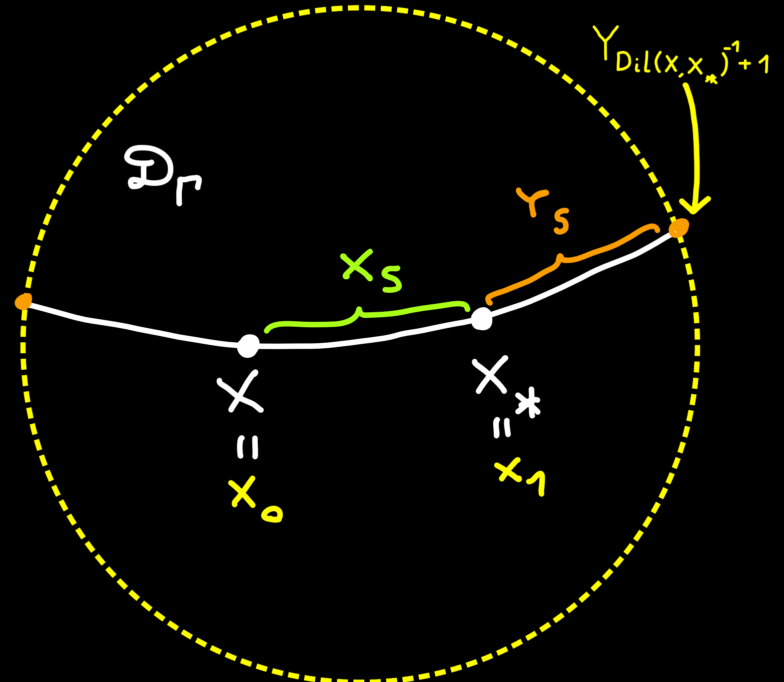
(Lang) $\exists$ $(\Gamma, d_t) \xhookrightarrow{\text{equivariant}} X_t$ & $[X_t] \in \mathfrak{D}_\Gamma$

$\hookrightarrow$ $s \in [1, \mathrm{Dil}(X,X_*)^{-1}+1) \rightsquigarrow$ $\boxed{\tilde{d}_s := d_* - (s-1)\, d}$

uniformly close to metric d_s on $\Gamma \rightsquigarrow$ get $[Y_s] \in \mathfrak{D}_\Gamma$

$\hookrightarrow$ Union $\{X_t\}_{0 \le t \le 1} \cup \{Y_s\}_{1 \le s < \mathrm{Dil}(X,X_*)^{-1}+1}$

is geodesic ray $\square$



RMK: Can use these geodesics to define a boundary $\partial_M \mathfrak{D}_\Gamma$

Thank You!!!