Cubulated Relatively Hyperbolic Groups

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Motivation & Main Result

Thm (Agol '13): G f.g. group

G cubulated + hyperbolic $\Longrightarrow$ G virtually special

Main Result (O.-R., Groves-Manning '20):

G cubulated + relatively hyperbolic

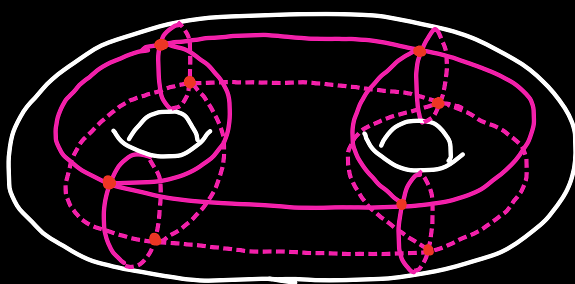
G virtually special $\Longleftrightarrow$ Peripheral subgroups (*) virtually special

Coro: cubulated + hyp. rel. virt. abelian $\Longrightarrow$ virt special

BACKGROUND

- CUBULATED GROUP: acts $\underbrace{\text{geometrically}}_{\substack{\text{proper + cocompact} \\ \text{by cubical isometries}}}$ on a CAT(0) cube complex

- EXAMPLES: Free groups, surface groups, RAAGs, π_1(fin-vol. hyp. 3-mfd), 1-relator groups w. torsion, some infinite simple groups



- (COCOMPACT) SPECIAL GROUPS

Combinatorially nice subgroups of RAAGs:

Γ graph $\rightsquigarrow A_\Gamma = \langle V(\Gamma) \mid [v,w]=1 \text{ iff } \{v,w\} \in E(\Gamma) \rangle$

Why is Specialness Useful?

Virtually special groups are

- Residually finite, virtually RFRS, convex cocompact subgroups are separable
- Schreve '14: Satisfy strong Atiyah conjecture
- Genevois '17: Hyperbolic $\longleftrightarrow$ No $\mathbb{Z}^2$ subgroups
  Hyp.rel.virt.abelian $\longleftrightarrow$ No $\mathbb{Z}\times F_2$ subgroups
- Agol '18: Embed into some compact Lie group
- Shepherd '22: $A_\Gamma \curvearrowright \widetilde{X}_\Gamma$, $G \curvearrowright \widetilde{X}_\Gamma$ cocompact lattice (X_Γ = Salvetti complex)

  G, A_Γ commensurable (up to finite index) in $\mathrm{Aut}(\widetilde{X}_\Gamma)$ $\longleftrightarrow$ $G \curvearrowright \widetilde{X}_\Gamma$ virtually special

Related Results

- Wise'96, Burger-Mozes '96,'00, Jankiewicz-Wise '17: ⇝ Non-Virtually special examples (Relatively Hyperbolic)
- Wise'11: π_1(cusped hyp 3-mfd), fully res. free groups are v. special
- Przytycki-Wise'13: M either a mixed 3-mfd or a graph mfd with boundary $\Rightarrow \pi_1(M)$ virt. special
- Groves-Manning'18: Hyperbolic + weakly cubulated $\Rightarrow$ v. special
- Einstein-Groves '20,'21: similar results for relatively geometric actions
- Hagen'11: Cubulated groups are weakly hyperbolic
- Conjecture: cubulated groups are hierarchically hyperbolic

MAIN RESULT: G cubulated + relatively hyperbolic

G virtually special $\iff$ Peripheral subgroups virtually special

+ combined with

- MARTIN-STEENBOCK '17: JANKIEWICZ-WISE '17 STUCKY '18

can construct rel. hyp. v. special groups wit prescribed (virt. special) peripheral subgroups

- EXAMPLE (GROVES-MANNING):

$$G = \langle a_1, \ldots, a_6, b_1, \ldots, b_6 \mid [a_1,b_1], \ldots, [a_6,b_6],\ a_1a_2a_3a_4a_5a_6,\ b_1b_2b_3b_4b_5b_6,\ a_4b_2b_3a_3a_1,\ a_3a_5b_1b_4a_6 \rangle$$

hyp. rel. virt. abelian & virt. special

SKETCH OF PROOF

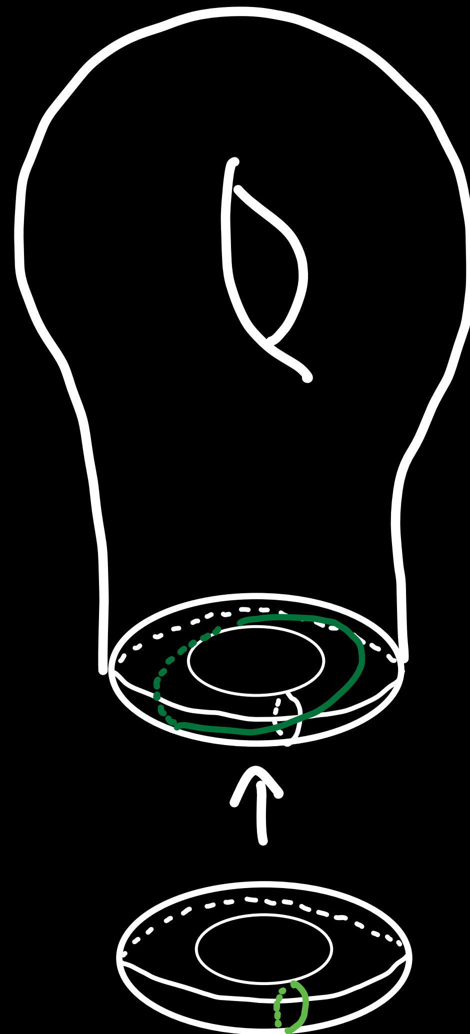
* **GOAL**: In Agol's proof, replace HYPERBOLIC with RELATIVELY HYPERBOLIC!!!

* **TOOL**: Group Theoretic Dehn Filling (OSIN, GROVES-MANNING)

$$(G, \{P_1, \ldots, P_n\}) \text{ rel. hyp.}$$

$$\Big\Downarrow \{N_i \triangleleft P_i\}_i$$

$$G \twoheadrightarrow \overline{G} = G / \langle\langle \cup_i N_i \rangle\rangle_G$$



"For most $\{N_i \overset{\text{fin. index}}{\triangleleft} P_i\}_i$, if G is nice, then $\overline{G}$ is nice & hyperbolic"

MSQT (Wise '11, Einstein '19):

G rel. hyperbolic & virt. special $\Longrightarrow$ $\overline{G}$ hyperbolic & virt. special

Relative Quasiconvex Hierarchy Theorem (Wise '11, O.-R. '20): G cubulated, rel. hyp. w virt special peripherals

$G = \pi_1$ of graph of groups
↳ convex edge/vertex group
↳ virtually special vertex groups $\Longrightarrow$ G virt. special
↳ satisfying a relative malnormality assumption

QUESTION

X finite special cube complex
Y finite NPC cube complex

$$X \underset{h.e.}{\simeq} Y$$

Is Y virtually special??

True if
- $\pi_1(X)$ hyperbolic
- $\pi_1(X)$ virt. abelian

What about $\pi_1(X)$ a RAAG?

Thank You!!!