Green metrics on Hyperbolic Groups and Reparameterizations of the Geodesic Flow

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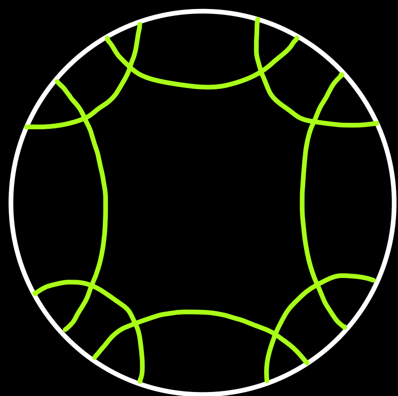
Motivation

(Σ, g) : closed negatively curved manifold $\quad \Gamma = \pi_1(\Sigma)$

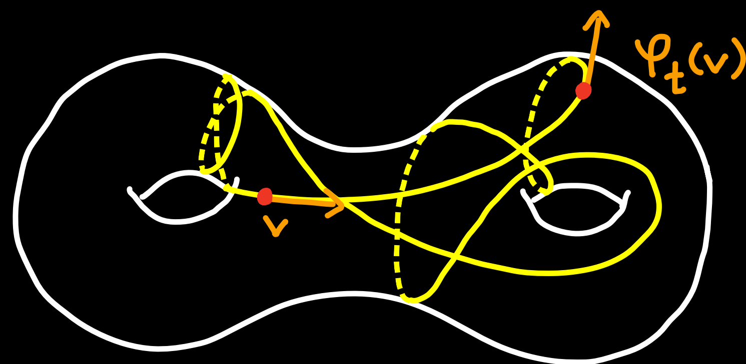
Isometric Action

$\Gamma \curvearrowright (\widetilde{\Sigma, g})$



Geodesic Flow

$\varphi_t = \varphi_t^g : T^1\Sigma \circlearrowleft$



Conjugacy classes in Γ		Oriented closed geodesics in (Σ, g)		Periodic orbits under geodesic flow
$[g]$	$\longleftrightarrow$	γ_g	$\longleftrightarrow$	$T^1\gamma_g$

Length Function: $\ell_g : \Gamma \longrightarrow \mathbb{R} \qquad g \longmapsto$ length of γ_g in Σ_g

Consider g_* another negatively curved metric on Σ

Geometric Action

$\Gamma \curvearrowright (\widetilde{\Sigma, g_*})$

isometric
proper
cobounded

Flow Reparameterization

$\exists\, \Psi : (T^1\Sigma, \varphi^g_\cdot) \longrightarrow (T^1\Sigma, \varphi^{g_*}_\cdot)$

homeo, orbit-preserving

$\tau^{g_*}_t := \Psi^{-1} \circ \varphi^{g_*}_t \circ \Psi$ flow on $T^1\Sigma$

Duality: $\ell_{g_*}(g) = \ell_{\tau^{g_*}}(g) := \tau^{g_*}$-period of $\gamma_g \quad \forall g \in \Gamma$

Theorem (Cantrell-Martinez-Granado-R. '25):

"Dictionary" $\left\{ \begin{array}{c} \text{Geometric actions} \\ \Gamma \curvearrowright X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Flow reparameterizations} \\ \tau = (\tau_t)_t : T^1\Sigma \circlearrowleft \end{array} \right\}$

↑ Gromov hyperbolic metric space

Length Functions

$\Gamma \curvearrowright X$ isometric action $\rightsquigarrow$ $\ell_X : \Gamma \longrightarrow \mathbb{R}$ $\quad g \longmapsto \lim_{k\to\infty} \frac{1}{k} d_X(x, g^k \cdot x)$

Examples:

* $S \subset \Gamma$ finite symmetric generating set

  $\rightsquigarrow$ $\Gamma \curvearrowright \mathcal{C}ay(\Gamma, S)$ $\rightsquigarrow$ $\ell_S : \Gamma \longrightarrow \mathbb{R}$

* $\Gamma = \pi_1(\overline{X})$, $\overline{X}$ compact NPC space

  $\rightsquigarrow$ $\Gamma \curvearrowright X$ geometric $\rightsquigarrow$ $\ell_X : \Gamma \rightarrow \mathbb{R}$

  (X: universal cover)

* $\Gamma \overset{\rho}{\curvearrowright} \mathbb{H}^n$ convex cocompact rep.

  $\Rightarrow \Gamma \overset{\rho}{\curvearrowright} \mathrm{Hull}(\Lambda(\rho(\Gamma)))$ geometric $\rightsquigarrow$ $\ell_\rho : \Gamma \longrightarrow \mathbb{R}$

Theorem (Furman '02): Γ hyperbolic, X, Y geodesic, $\Gamma \curvearrowright X, \Gamma \curvearrowright X$ geometric

$\ell_X(g) = \ell_Y(g) \ \forall g \in \Gamma$ $\longleftrightarrow$ X, Y Γ-equivariantly almost isometric

Main Result

Theorem (C-MG-R '25): $\Gamma = \pi_1(\Sigma)$, (Σ, g) closed neg. curved mfd

1) $\Gamma \curvearrowright X$ geometric action $\Rightarrow$ $\exists$ continuous flow τ on $T^1\Sigma$, orbit equivalent to φ^g,

s.t. $\ell_X(g) = \ell_\tau(g) \quad \forall g \in \Gamma$

New for $X = \mathcal{C}ay(\Gamma, S)$, $\mathcal{X}$ CAT(0) cube complex!

2) $\tau = \tau_t : T^1\Sigma \circlearrowleft$ Hölder flow orbit equivalent to φ^g $\Rightarrow$ $\exists$ $\Gamma \curvearrowright X$ geometric action (possibly assymetric)

s.t. $\ell_X(g) = \ell_\tau(g) \quad \forall g \in \Gamma$

[done by Connell-Muchnik '07]

Main Tool: Green Metrics

Γ finitely generated group

- $\lambda \in \mathrm{Prob}(\Gamma)$ **admissible**:
  - $\mathrm{supp}(\lambda)$ finite & generates Γ
  - $\lambda(g^{-1}) = \lambda(g) \ \forall g$

$\Rightarrow$ **Random walk** $(Z_n)_n$ on Γ: $Z_n = X_1 \cdot \ldots \cdot X_n$ X_i: i.i.d. random variables with law λ

- **Green Metric (on Γ)**: $d_\lambda(g,h) := -\mathrm{Log}\,\mathbb{P}(\exists n \mid g Z_n = h)$

Properties

$$d_\lambda(g,h) := -\operatorname{Log} \mathbb{P}(\exists n \mid gZ_n = h)$$

* Γ non-amenable $\Rightarrow$ d_λ quasi-isometric to word metric

* **Theorem (Blachère-Haïssinsky-Mathieu '11):**

  Γ non-elementary hyperbolic $\Rightarrow$ d_λ δ-hyperbolic

* **Theorem (Ledrappier '95, Nica-Špakula '15):**

  $\Gamma = \pi_1(\Sigma)$, (Σ, g) closed neg. curved manifold

  $\Rightarrow \exists$ Hölder flow reparameterization $\tau = (\tau_t)_t$ of $T^1\Sigma$ dual to d_λ

  almost CAT(-1)!

Density of Green Metrics

Theorem (C-MG-R '25): Γ non-elementary hyperbolic group $\Gamma \curvearrowright X$ geometric $\Rightarrow$ $\exists$ sequence $\lambda_k \in Prob(\Gamma)$ admissible s.t. $\ell_{\lambda_k} \longrightarrow \ell_X$ "uniformly"

($\exists\, r>0$ s.t. if $x \in X$ $\Rightarrow$ $\lambda_k :=$ uniform probability measure with support $S_k := \{g \in \Gamma : |d_X(x, gx) - k| \leq r\}$)

Corollary (Gouëzel-Mathéus-Macourant '15):

$\ell_{\lambda_k} \longrightarrow \ell_X$ "on average"

Proof of Main Result

Start with $\Gamma \curvearrowright X$ geometric

↳ Find Green metrics d_{λ_k} s.t. $\ell_{\lambda_k} \to \ell_X$ uniformly

↳ Each d_{λ_k} is dual to a reparameterization $\tau^k = (\tau^k_t)_t : T^1\Sigma \circlearrowleft$

↳ Up to conjugacies & subsequences, have $\tau^\infty = (\tau^\infty_t)_t : T^1\Sigma \circlearrowleft$

↳ Show τ^∞ is dual to $\Gamma \curvearrowright X$

Thank You!!!