Complejos Cúbicos y Curvatura no Positiva

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Coloquio UC

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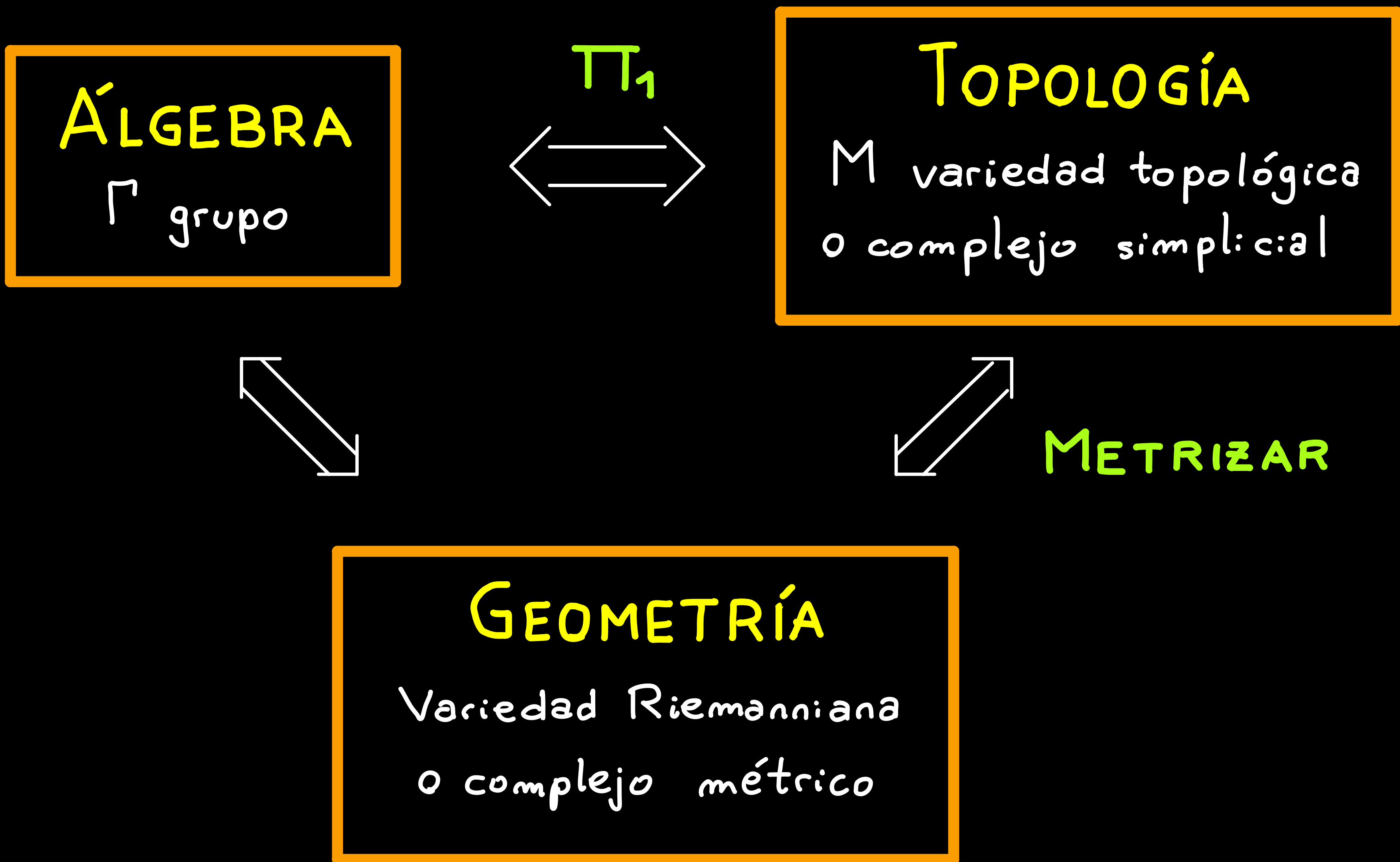
MOTIVACIÓN



HIPÓTESIS: CURVATURA NO POSITIVA

Recuerdo de Grupos

$\Gamma = (\Gamma, \cdot)$, $\quad \Gamma \times \Gamma \xrightarrow{\cdot} \Gamma$ asociativa, $\exists$ neutro e, $\exists$ inversos

Ejemplos (finitos): $\{e\}$, $\mathbb{Z}/n\mathbb{Z}$, S_n, A_n, D_n, productos (semi)directos

Ejemplos (infinitos): $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$, $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n,\mathbb{R})$

Presentación: $\quad \Gamma = \langle S \mid R \rangle$

generadores $\quad$ relaciones

Ejemplos:

* $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$
* $D_n = \langle r, s \mid r^n, s^2, srs^{-1}r \rangle$
* $F_n = \langle x_1, \dots, x_n \mid \emptyset \rangle$ Grupo Libre

RECUERDO DE GRUPOS

$\Gamma = (\Gamma, \cdot)$, $\Gamma \times \Gamma \xrightarrow{\cdot} \Gamma$ asociativa, $\exists$ neutro e , $\exists$ inversos

EJEMPLOS (FINITOS): $\{e\}$, $\mathbb{Z}/n\mathbb{Z}$, S_n, A_n, D_n, productos (semi) directos

EJEMPLOS (INFINITOS): $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{R}^n$, $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n,\mathbb{R})$

HOY: GRUPOS DISCRETOS INFINITOS

PRESENTACIÓN: $\Gamma = \langle S \mid R \rangle$

generadores relaciones

$$\boxed{\begin{array}{ll} S \text{ finito} & (\text{FINITAMENTE GENERADO}) \\ S, R \text{ finitos} & (\text{FINITAMENTE PRESENTADO}) \end{array}}$$

EJEMPLOS:

* $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$
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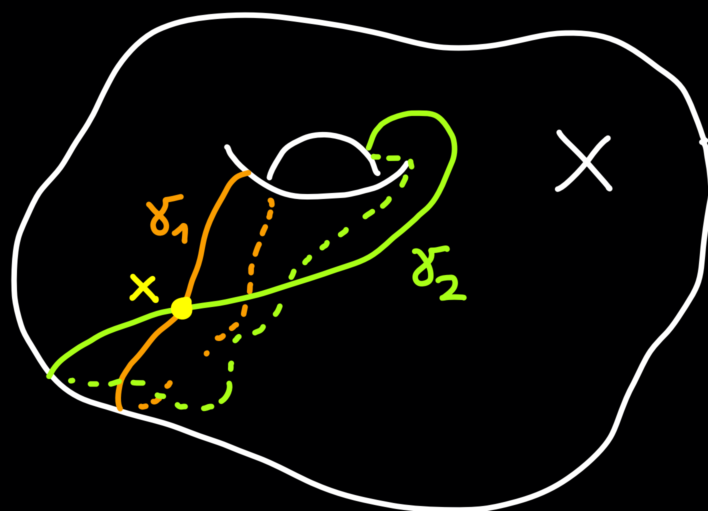
Grupo Fundamental

X espacio topológico arcoconexo, $x \in X$

$$\boxed{\pi_1(X) = \left\{ \text{Curvas } \gamma : [0,1] \to X,\ \gamma(0) = \gamma(1) = x \right\} \Big/ \text{homotopía}}$$

OPERACIÓN: Concatenar caminos

$$\gamma_1 * \gamma_2$$



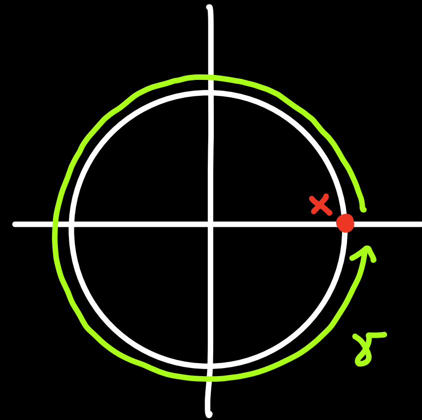
$\pi_1(X)$ ES GRUPO!

$$[f] * [g] := [f * g]$$

NEUTRO : $e = [\text{constante} = x]$

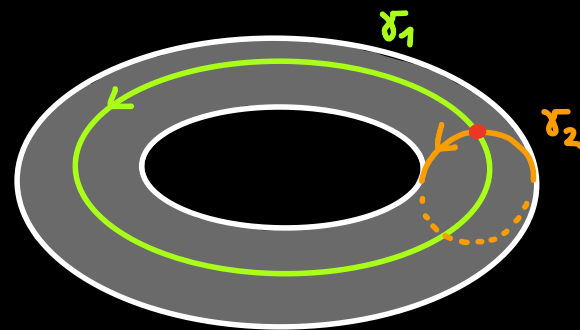
INVERSA: $[\gamma]^{-1} = [\bar{\gamma}]$ revertir orientación

Ejemplos



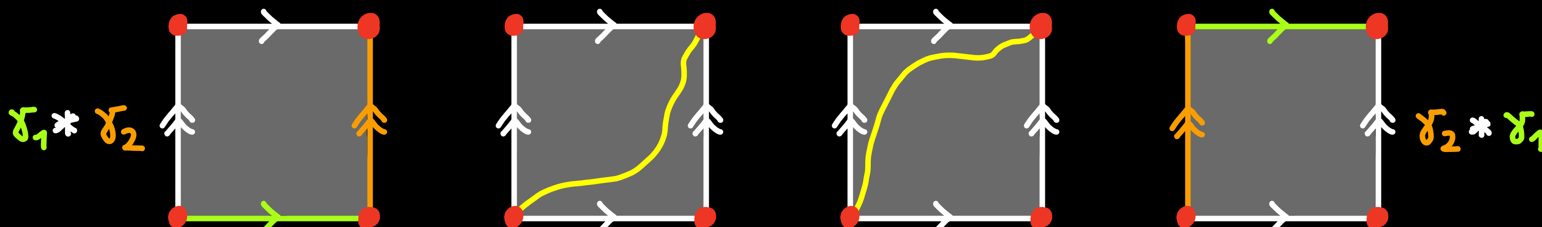
* **Círculo** $\quad \mathbb{S}^1=\{z\in\mathbb{C} \mid |z|=1\}$

$$\pi_1(\mathbb{S}^1)\simeq \mathbb{Z} = \langle[\gamma]\rangle, \quad \gamma(t)=e^{2\pi i t}$$



* **2-Toro** $\quad \mathbb{T}^2=\mathbb{S}^1\times\mathbb{S}^1$

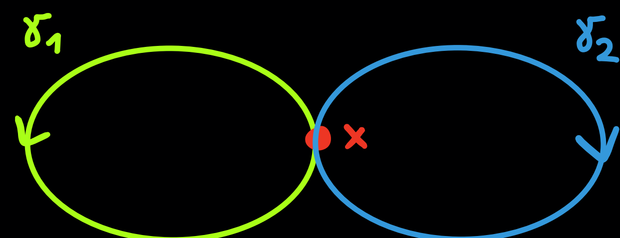
$$\pi_1(\mathbb{T}^2)\simeq \pi_1(\mathbb{S}^1\times\mathbb{S}^1)\simeq\mathbb{Z}^2=\langle[\gamma_1],[\gamma_2]\rangle$$



$$[\gamma_1]*[\gamma_2] = [\gamma_2]*[\gamma_1]$$

* **Rosa** $\quad \mathcal{R}_2=\mathbb{S}^1\vee\mathbb{S}^1$

$$\pi_1(\mathcal{R}_2)\simeq F_2=\langle[\gamma_1],[\gamma_2]\rangle$$



CURVATURA NO POSITIVA (RIEMANNIANA)

M^n variedad Riemanniana suave, $\Gamma \simeq \pi_1(M)$

- Compacta, sin borde, orientable
- Curvaturas seccionales ≤ 0

$\Longrightarrow$

* Γ es finitamente presentado, infinito, sin torsión
* Cubrimiento universal $\tilde{M}$ es $\approx \mathbb{R}^n$ ($\therefore$ contractible)
* Si $H < \Gamma$ isomorfo a $\mathbb{Z}^k$

  $\Rightarrow \exists$ k-toro $\mathbb{T}^k \overset{f}{\hookrightarrow} M$ totalmente geodésico tal que

$$f_*(\pi_1(\mathbb{T}^k)) = H$$

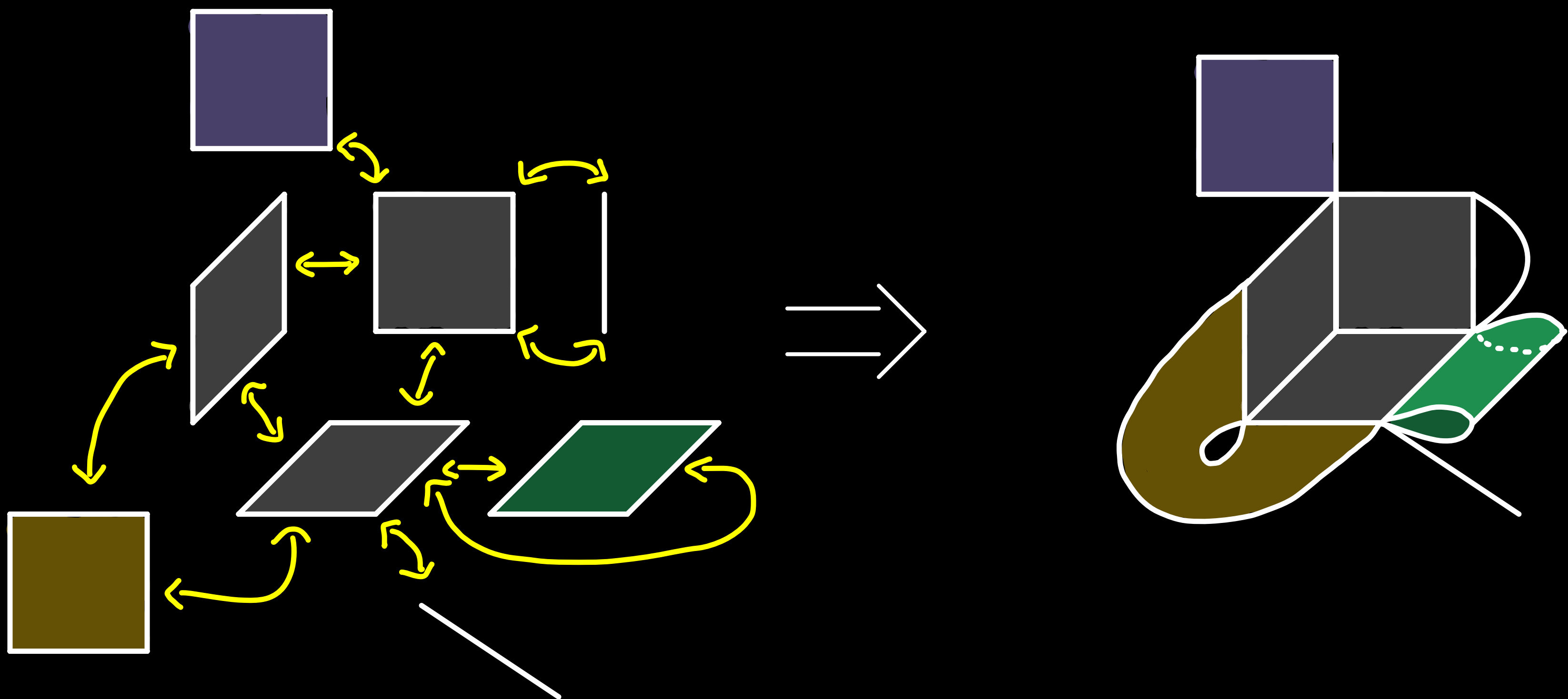
IDEA (GROMOV '80s): Γ grupo de interés

$\Longrightarrow$ Ver $\Gamma \simeq \pi_1(X)$

X no variedad, pero con CURVATURA NO POSITIVA

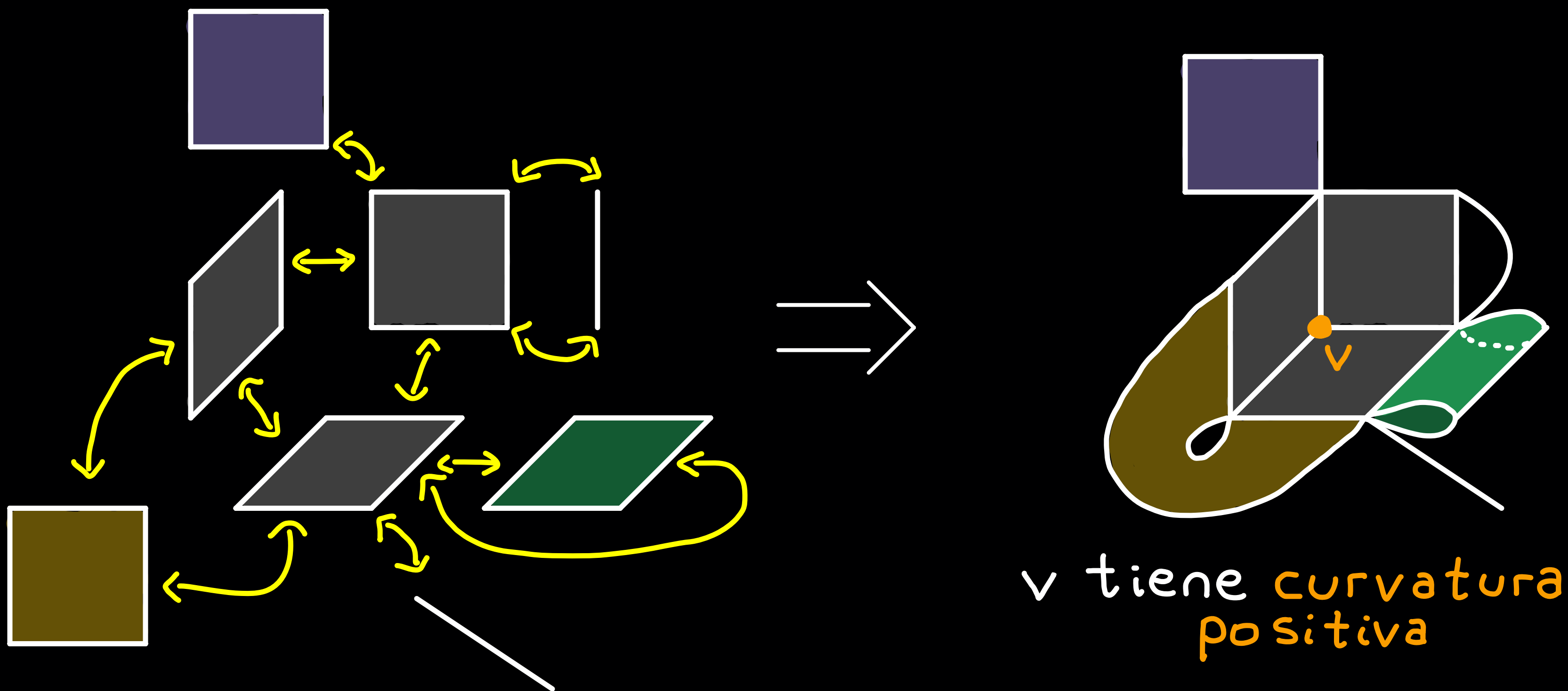
Complejos Cúbicos NPC

Complejo Cúbico: Pegar cubos Euclídeos de lado 1 isométricamente por (sub)caras



COMPLEJOS CÚBICOS NPC

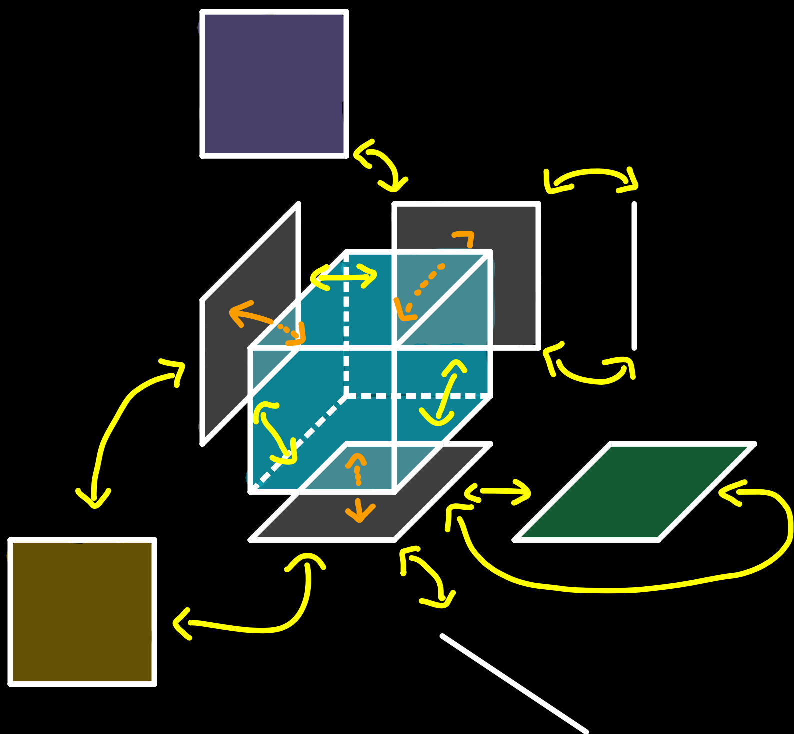
COMPLEJO CÚBICO: Pegar cubos Euclídeos de lado 1 isométricamente por (sub)caras

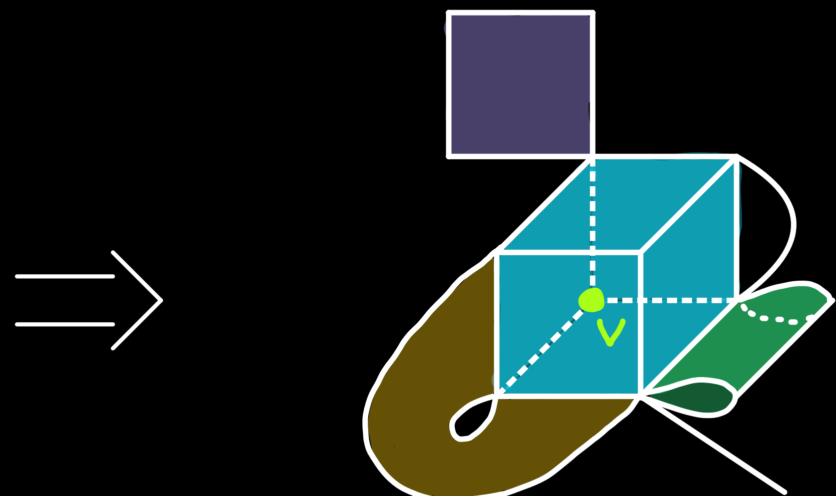


NPC: Condición de curvatura en cada vértice: "**NO ESFERAS**"

Complejos Cúbicos NPC

Complejo Cúbico: Pegar cubos Euclídeos de lado 1 isométricamente por (sub)caras

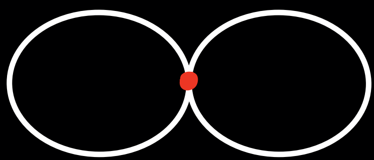


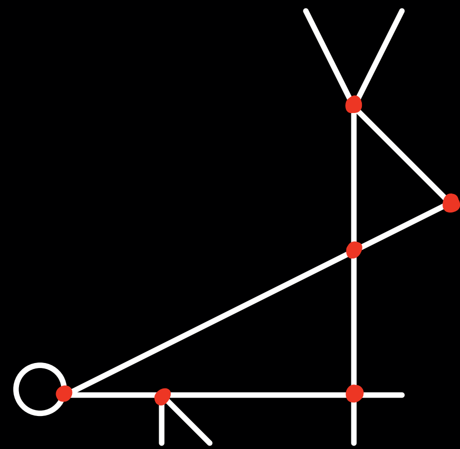


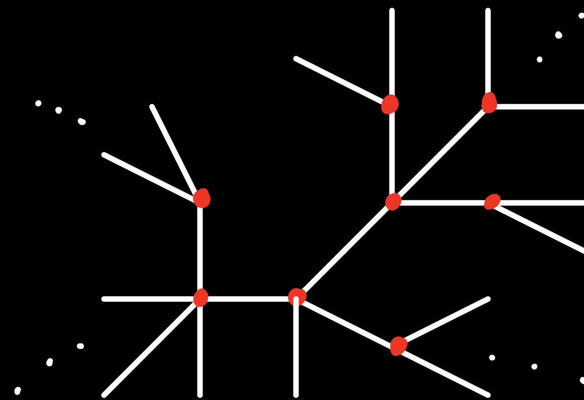
NPC: Condición de curvatura en cada vértice: "**NO ESFERAS**"

Ejemplos

* **Grafos**

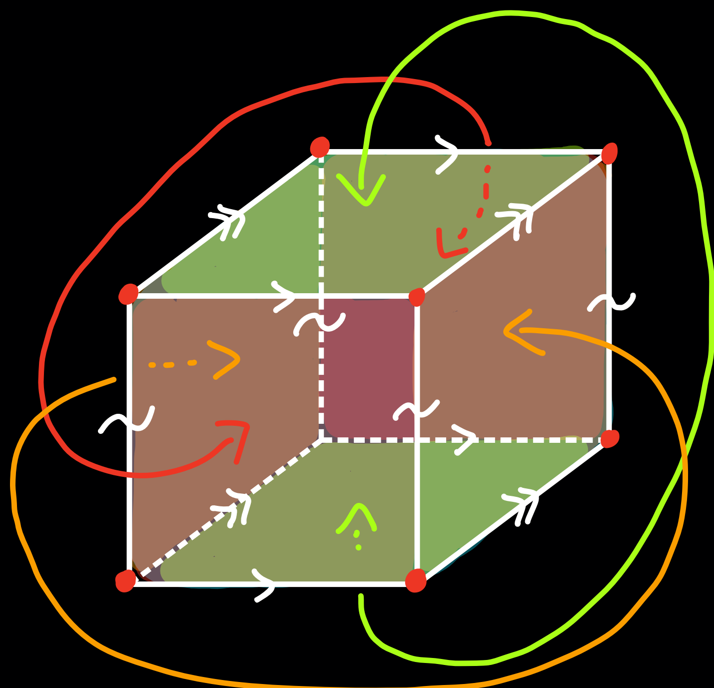






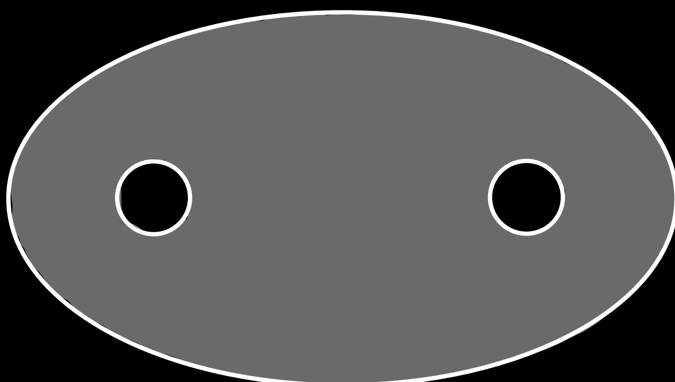
* **n- Toros**

$\mathbb{T}^3 =$



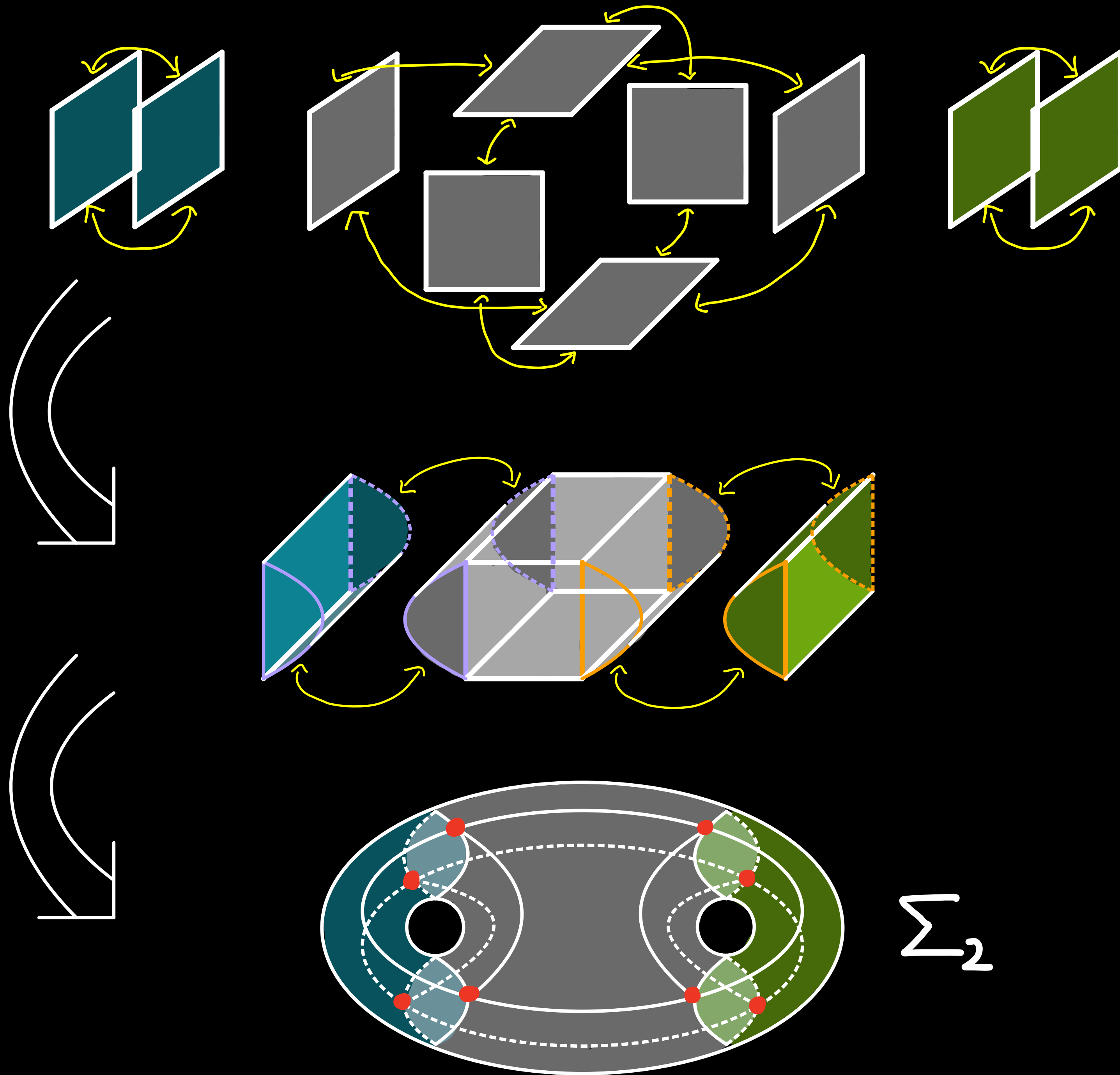
$= \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{T}^1$

Ejemplo: Superficies



Σ_2

Ejemplo: Superficies

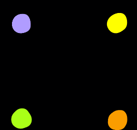
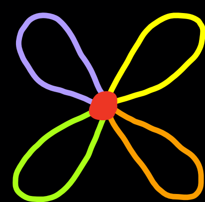


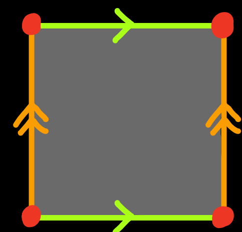
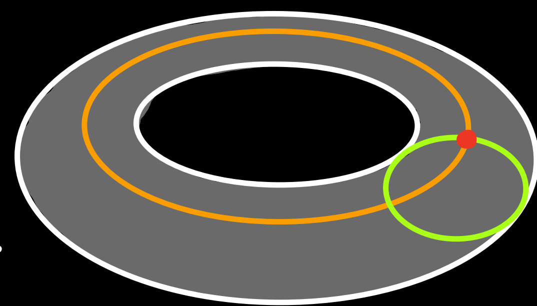
Σ_2

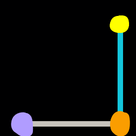
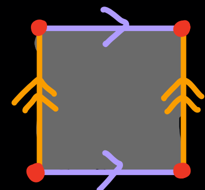
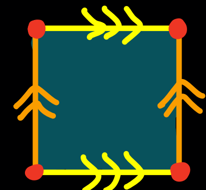
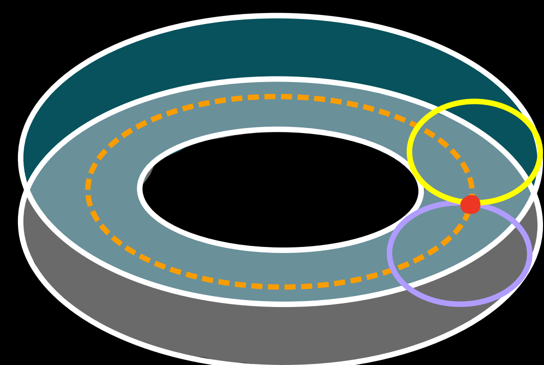
Ejemplo: Complejos de Salvetti

$$\mathcal{X}_G := \begin{cases} \,1 \quad \text{vértice} \\ +\,1 \quad 1\text{-toro} \quad \text{por cada} \quad \text{vértice de } G \\ +\,1 \quad 2\text{-toro} \quad \text{por cada} \quad \text{arista de } G \\ \quad\quad \vdots \\ +\,1 \quad n\text{-toro} \quad \text{por cada} \quad n\text{-grafo completo de } G \\ \quad\quad \vdots \end{cases}$$

grafo simplicial finito

$G =$  $\rightsquigarrow$ $\mathcal{X}_G = \mathcal{R}_4 =$ 

$G =$  $\rightsquigarrow$ $\mathcal{X}_G = \mathbb{T}^2 =$  $=$ 

$G =$  $\rightsquigarrow$ $\mathcal{X}_G =$  $\cup$  $=$ 

Grupos Cubulables

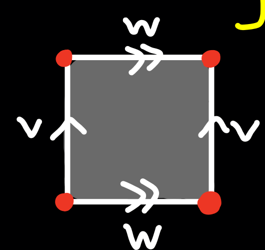
Γ grupo **CUBULABLE**: $\Gamma \simeq \pi_1^{orb}(\mathcal{X})$

$\mathcal{X}$ complejo cúbico NPC compacto

Ejemplos:

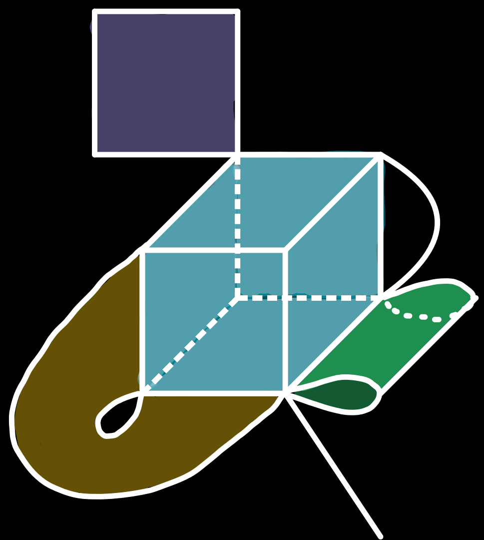
* $\mathbb{Z}^n \simeq \pi_1(\mathbb{T}^n)$ (cubulable × cubulable = cubulable)
* $F_n \simeq \pi_1(\mathcal{R}_n)$ (cubulable * cubulable = cubulable)
* Grupos de superficie: $\pi_1(\underbrace{\Sigma_g}_{\text{superficie de género } g}) = \langle a_1,b_1,\dots,a_g,b_g \mid [a_1,b_1]\cdot \ldots \cdot [a_g,b_g]\rangle$
* **RAAGs**: Grupos de Artin de ángulo recto

  $A_G = \pi_1(\mathcal{X}_G) = \left\langle \text{Vértices de } G \;\middle|\; vw=wv \text{ ssi } \{v,w\} \text{ arista de } G \right\rangle$



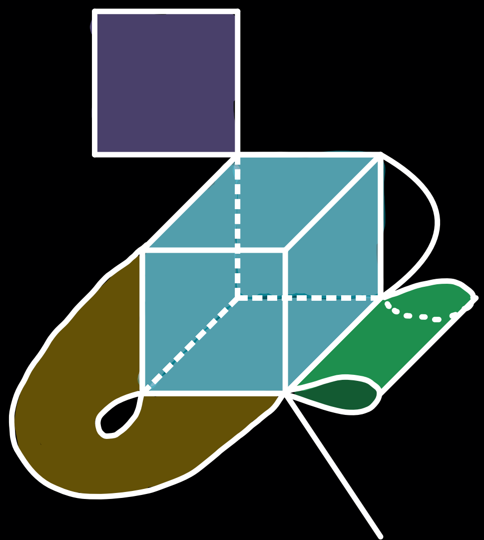
CÓMO CUBULAR UN GRUPO?

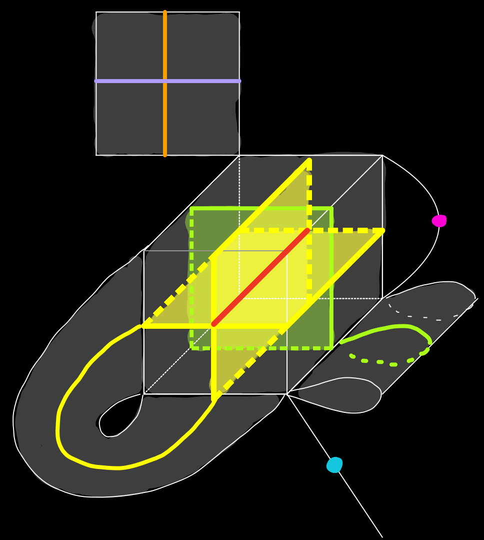
HIPERPLANOS $\longleftrightarrow$ SUBGRUPOS DE CODIMENSIÓN 1



Cómo cubular un grupo?

Hiperplanos ⟷ Subgrupos de codimensión 1

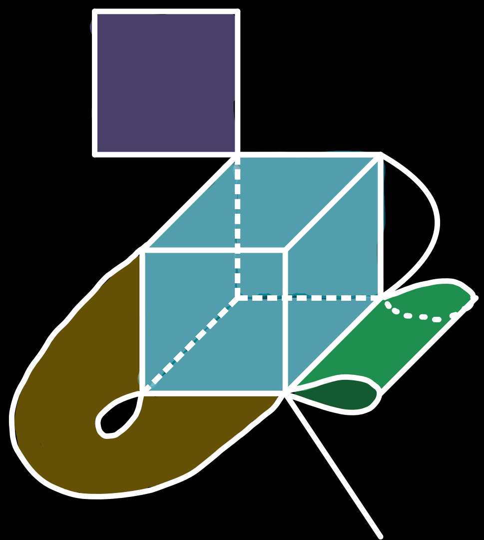


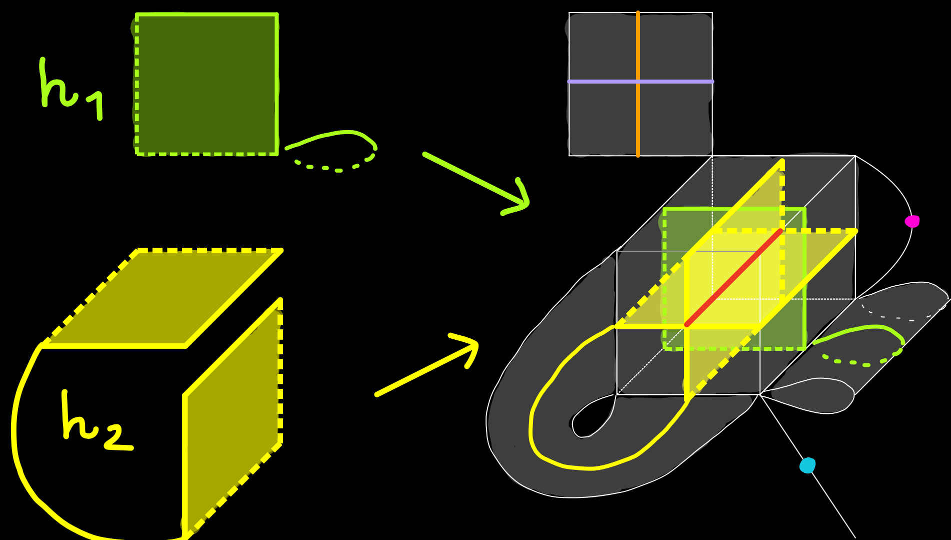


Cómo cubular un grupo?

HIPERPLANOS $\longleftrightarrow$ SUBGRUPOS DE CODIMENSIÓN 1

PROPIEDAD: $\pi_1(h) \hookrightarrow \pi_1(X)$





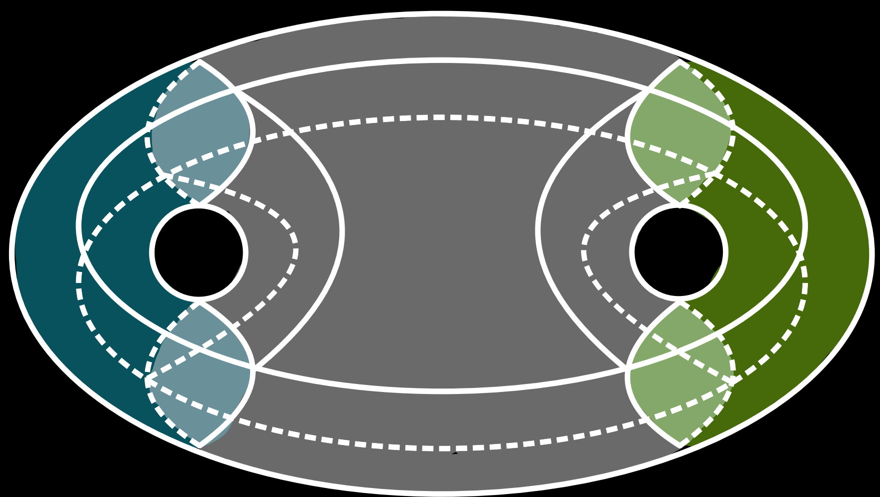
$\pi_1(h_1) \simeq \mathbb{Z}$

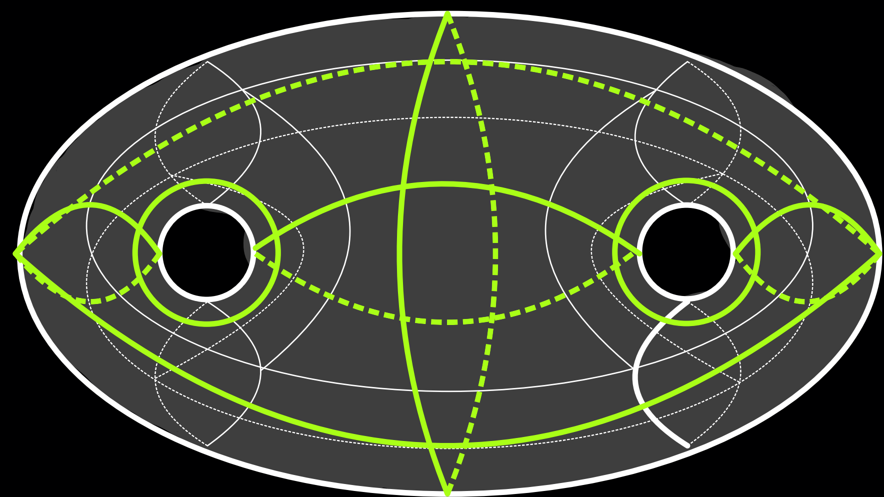
$\pi_1(h_2) \simeq \{e\}$

Cómo cubular un grupo?

Hiperplanos $\longleftrightarrow$ Subgrupos de codimensión 1

$$\boxed{\textbf{Propiedad: } \pi_1(h) \hookrightarrow \pi_1(X)}$$



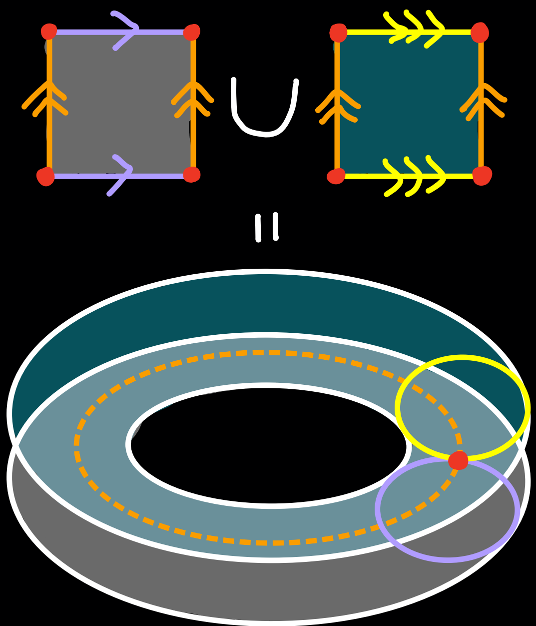


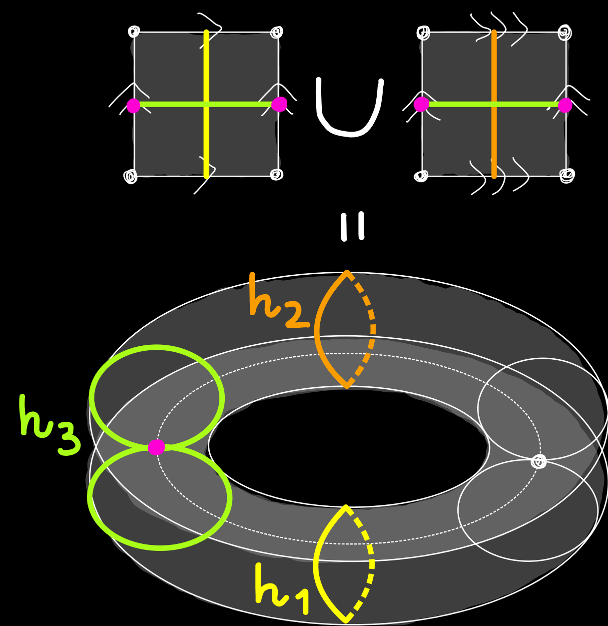
$\pi_1(h) \simeq \mathbb{Z} \quad \forall\, h$

Cómo cubular un grupo?

Hiperplanos $\longleftrightarrow$ Subgrupos de codimensión 1

$\boxed{\text{Propiedad: } \pi_1(h) \hookrightarrow \pi_1(X)}$





$\pi_1(h_1) \simeq \mathbb{Z}$

$\pi_1(h_2) \simeq \mathbb{Z}$

$\pi_1(h_3) \simeq F_2$

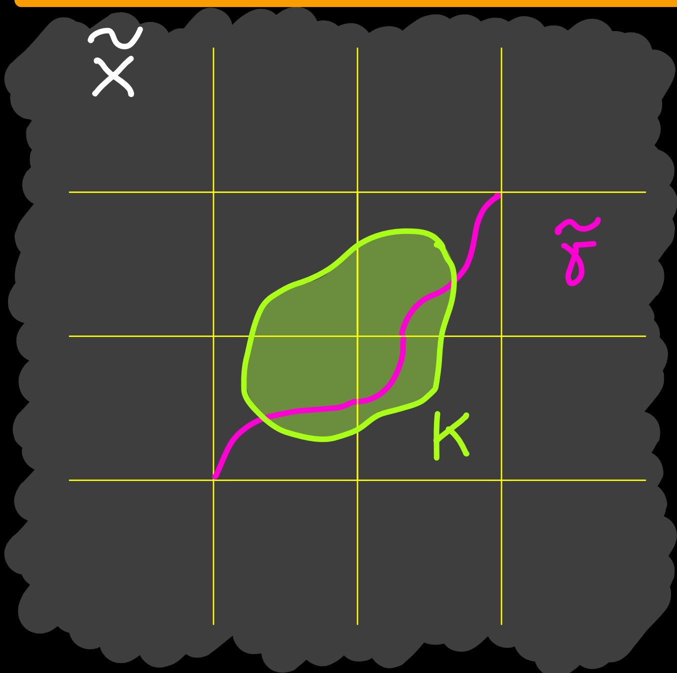
Para qué cubular?

Resolución de singularidades en cubrientes finitos

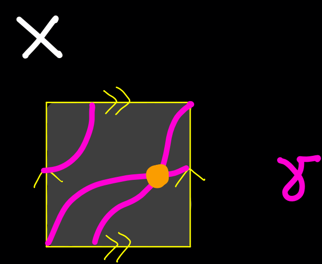
Propiedad: Todo RAAG A_G es **Residualmente Finito**

$A_G \cong \pi_1(X)$, $\tilde{X} \xrightarrow{p} X$ cubriente universal, $K \subset \tilde{X}$ compacto

CW complejo



$\xrightarrow{p}$



Para qué cubular?

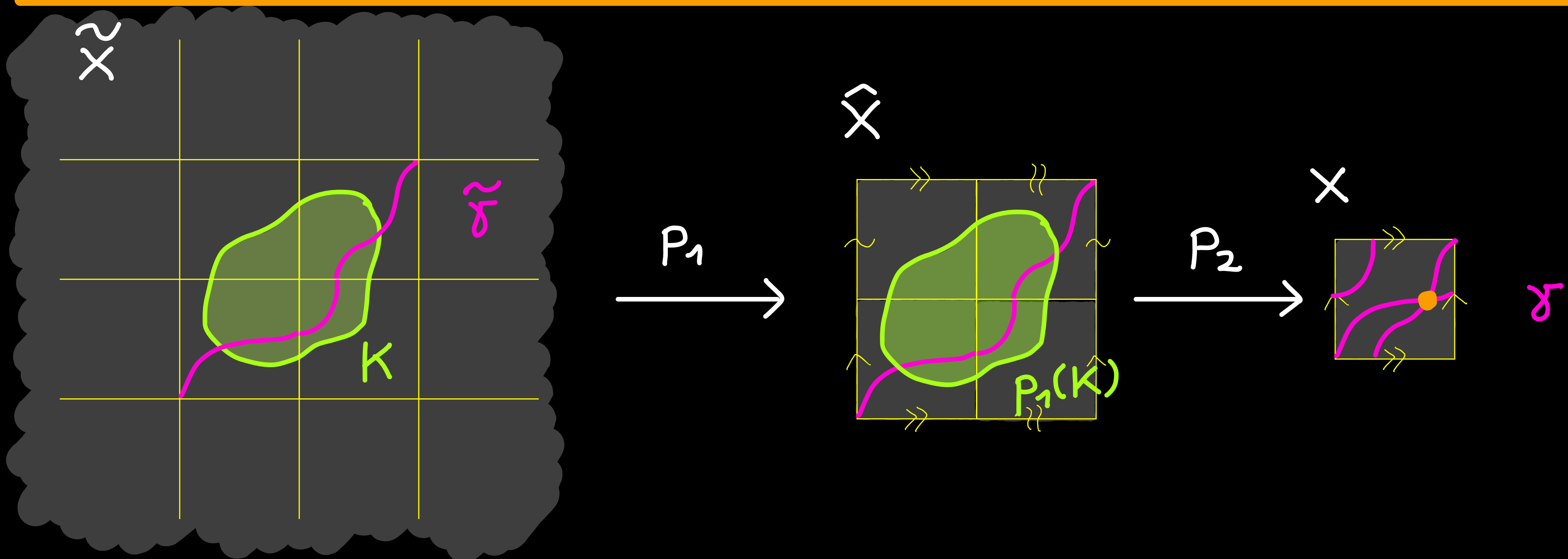
Resolución de singularidades
en cubrientes finitos

Propiedad: Todo RAAG A_G es **Residualmente Finito**

$A_G \cong \pi_1(X)$, $\tilde{X} \xrightarrow{p} X$ cubriente universal, $K \subset \tilde{X}$ compacto

CW complejo

$\Rightarrow \exists\ \tilde{X} \xrightarrow{p_1} \hat{X} \xrightarrow{p_2} X$, p_2 cubriente finito, $p_1|_K$ inyectiva



Equivalente: Topología profinita en A_G es Hausdorff

APLICACIÓN: GRUPOS DE UN RELATOR CON TORSIÓN

$$\Gamma = \langle s_1, \dots, s_n \mid r^m \rangle \quad m > 1$$

EJEMPLOS:
* $\langle a, b \mid b^m \rangle \simeq \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$
* $\langle a_1, \dots, a_{2025} \mid (a_1 a_3 a_5 \cdot \dots \cdot a_{2025})^3 \rangle$

TEOREMA (WISE '11): Γ grupo de 1 relator con torsión
$\Rightarrow$ 1) Γ cubulable : $\Gamma = \pi_1^{orb}(X)$
2) $\exists\, \Gamma' < \Gamma$, $|\Gamma : \Gamma'| < \infty$ tq $\Gamma' \hookrightarrow A_G \leftarrow$ RAAG

CONSECUENCIA: Γ es residualmente finito [Solución a Conjetura de Baumslag]

Aplicación: Grupos de 3-variedades

$\Gamma \simeq \pi_1(M)$, M 3-variedad Riemanniana, completa, sin borde
curvatura $\equiv -1$ ($\tilde{M} \underset{\text{isom}}{\simeq} \mathbb{H}^3$) + volumen finito

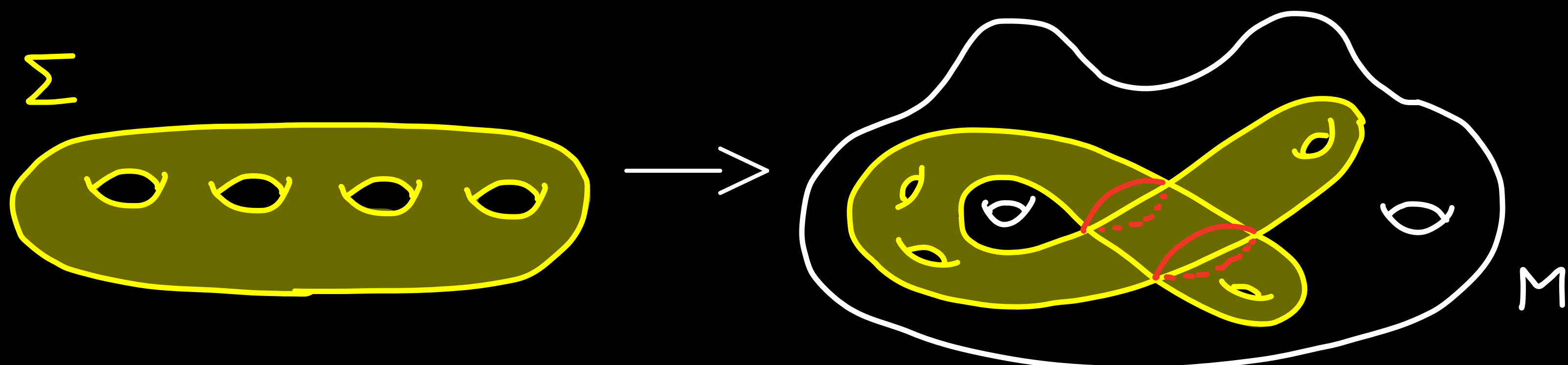
- 3-variedades "genéricas"
- Más misteriosas

Ejemplo: $\mathbb{S}^3 \setminus$ 

Teorema (Kahn-Markovic '12, Bergeron-Wise '11):
M compacta $\Rightarrow$ Γ cubulable

$\rightsquigarrow$ Existen muchas superficies π_1-inyectivas (codimensión 1)

$\Sigma \hookrightarrow M$
$\pi_1(\Sigma) \hookrightarrow \Gamma$

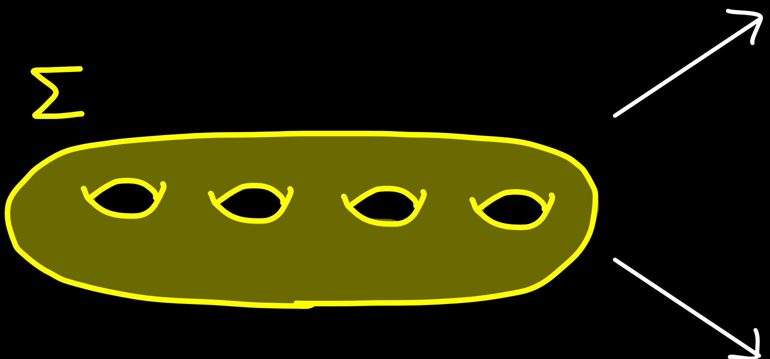


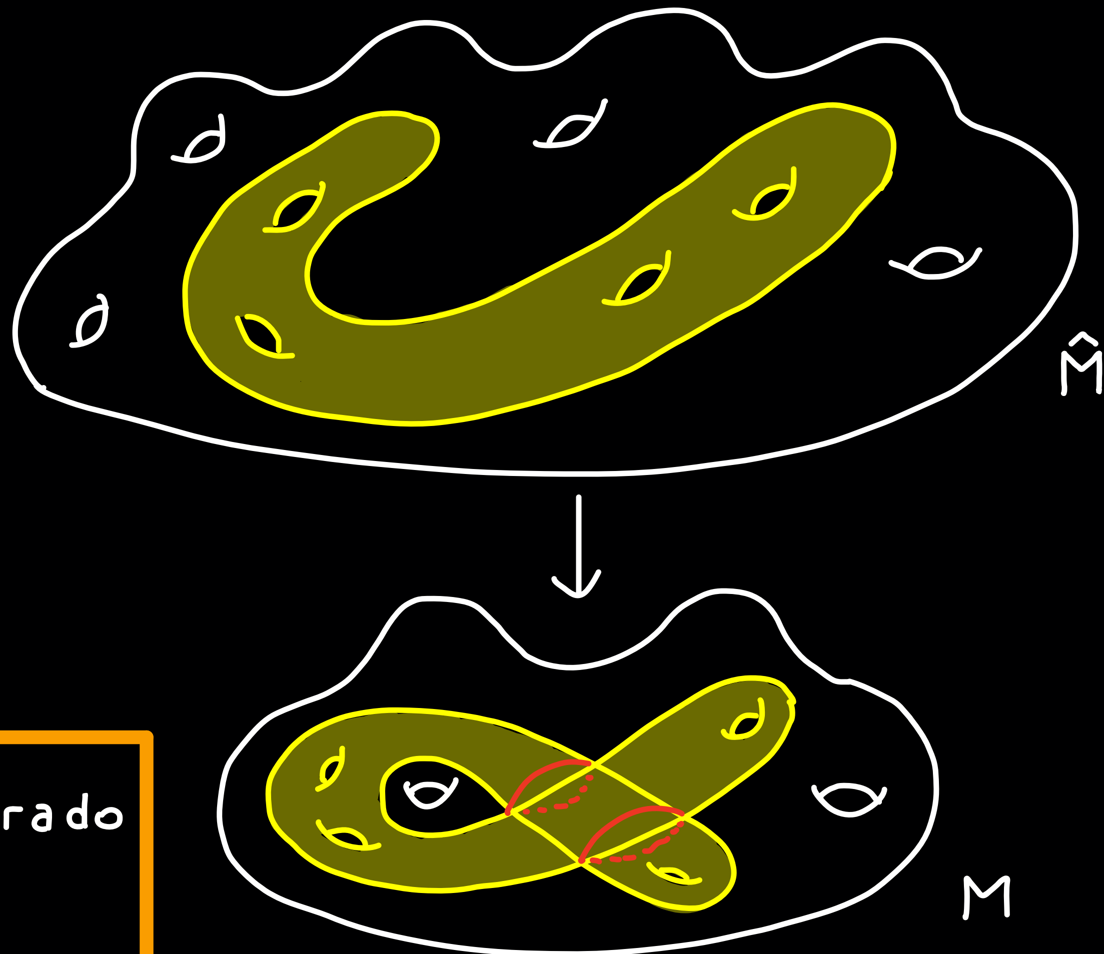
compacto no-compacto

Teorema (Agol'13, Wise'10): $\exists\, \Gamma' < \Gamma,\ |\Gamma : \Gamma'| < \infty,\ \Gamma' \hookrightarrow A_G$

Consecuencia: Existe cubriente finito $\hat{M} \to M$ tal que $\Sigma \hookrightarrow \tilde{M}$ incrustación

[Solución a Conjetura de Waldhausen]





Propiedad: $\pi_1(\Sigma)$ es cerrado en topología profinita de Γ

Extensiones de teoremas de Agol y Wise

Teorema (Groves-Manning '23, R. '23):

$$\Gamma = \underbrace{A_{G_1} * \dots * A_{G_n}}^{\text{Producto libre de RAAGs}} / \langle\langle r_1, \dots, r_k \rangle\rangle$$

cociente de "cancelación pequeña"

$\Longrightarrow$ Subgrupos "convexos" de Γ son cerrados en topológia profinita

Ejemplo (Groves-Manning):

$$\Gamma = \left\langle a_1, \dots, a_6, b_1, \dots, b_6 \;\middle|\; \begin{array}{l} [a_1, b_1], \dots, [a_6, b_6], \\ a_1 a_2 a_3 a_4 a_5 a_6,\ b_1 b_2 b_3 b_4 b_5 b_6 \\ a_4 b_2 b_5 a_3 a_1,\ a_3 b_5 b_1 b_4 a_6 \end{array} \right\rangle$$

FUTURO DE CUBULACIONES

1) CUBULAR MÁS Y MEJOR

PREGUNTA: M^n variedad hiperbólica compacta, $n \geq 4$

$\pi_1(M)$ cubulable?

PREGUNTA: $\underline{M^4 \text{ variedad atoroidal con fibración}}$ $\Sigma_{g_1} \to M \to \Sigma_{g_2}$

$g_1, g_2 \geq 2$

$\pi_1(M)$ cubulable?

Existen por
KENT-LEININGER '24

2) RELACIÓN CON OTRAS ESTRUCTURAS GEOMÉTRICAS

TEOREMA (DOUBA-FLÉCHELLES-WEISMAN-ZHU '23):

Γ hiperbólico + cubulable $\Longrightarrow$ Existe representación Anosov $\Gamma \to PSL_d(\mathbb{R})$

3) Genericidad de cubulaciones

Pregunta (Futer-Wise '21): Γ hiperbólico + cubulable

$\Gamma = \pi_1(X)$, X compacto $\Rightarrow \exists$ cubulaciones $\mathcal{X}_k$ con "$\mathcal{X}_k \xrightarrow{k\to\infty} X$"?

Teorema (Brody-R. '24): Cierto si $X = M$, 3-variedad hiperbólica

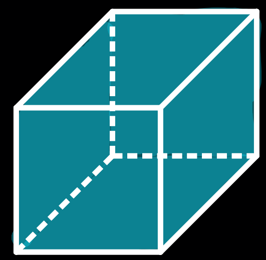
4) Conteo de curvas

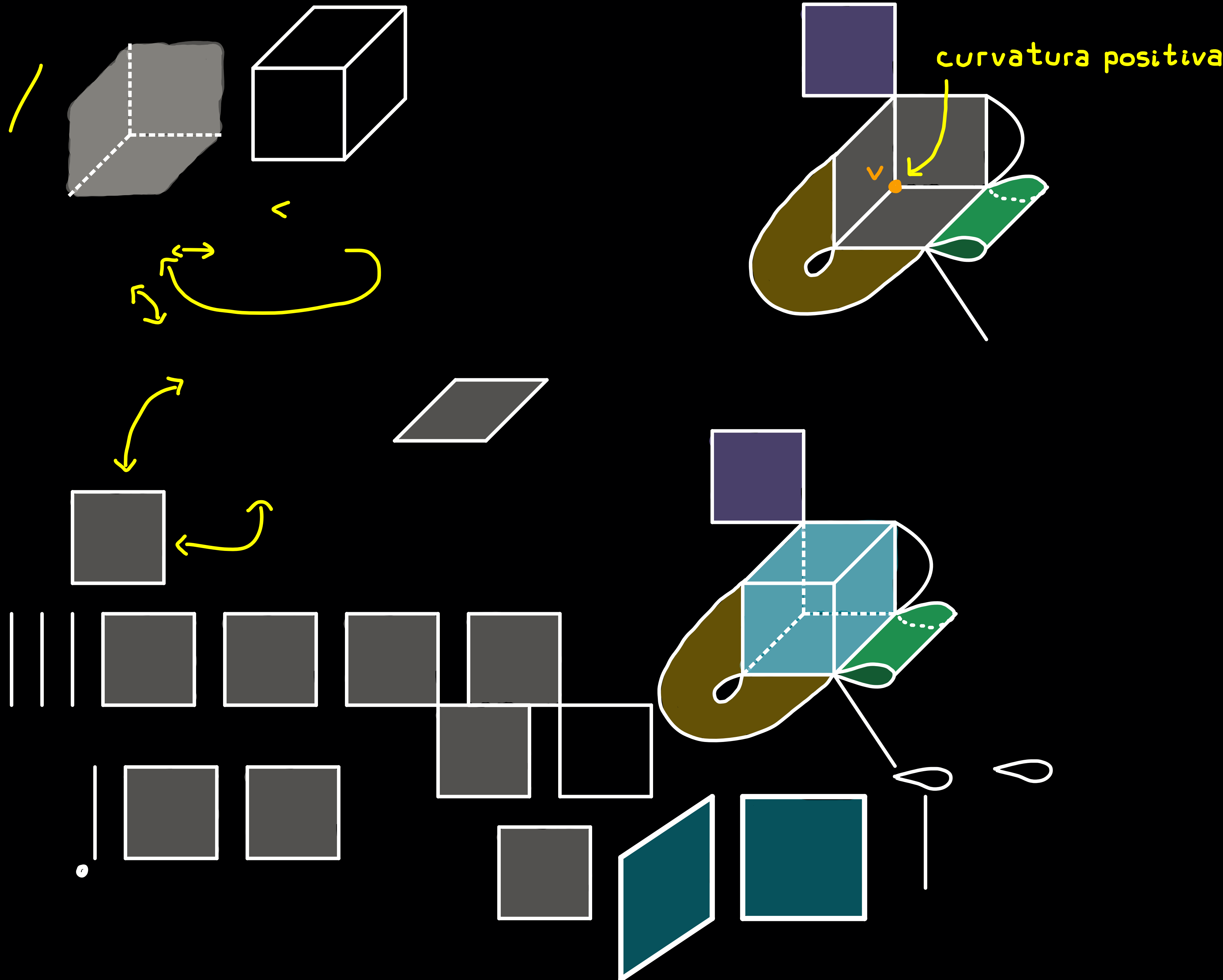
Teorema (Cantrell-R. '23): Γ hiperbólico, $\mathcal{X}$ cubulación de Γ

Entendimiento de

$$N_{\mathcal{X}}(T) = \#\left\{ \gamma \ \begin{array}{l}\text{geodésica}\\ \text{combinatorial}\\ \text{cerrada en } \mathcal{X}\end{array} \ \middle|\ \ell_{\mathcal{X}}(\gamma) < T \right\}$$

GRACIAS!!!

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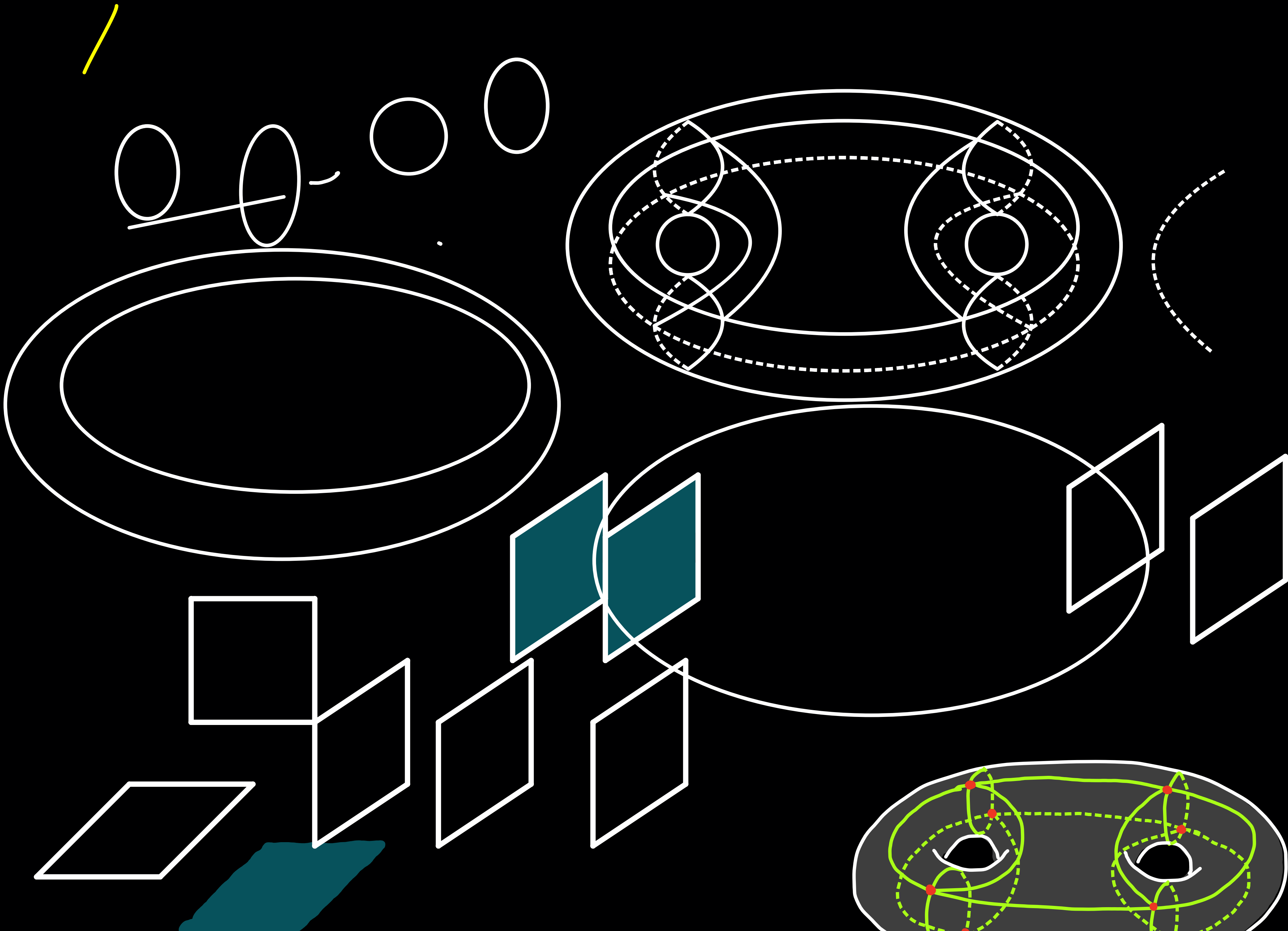


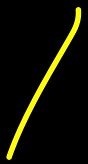
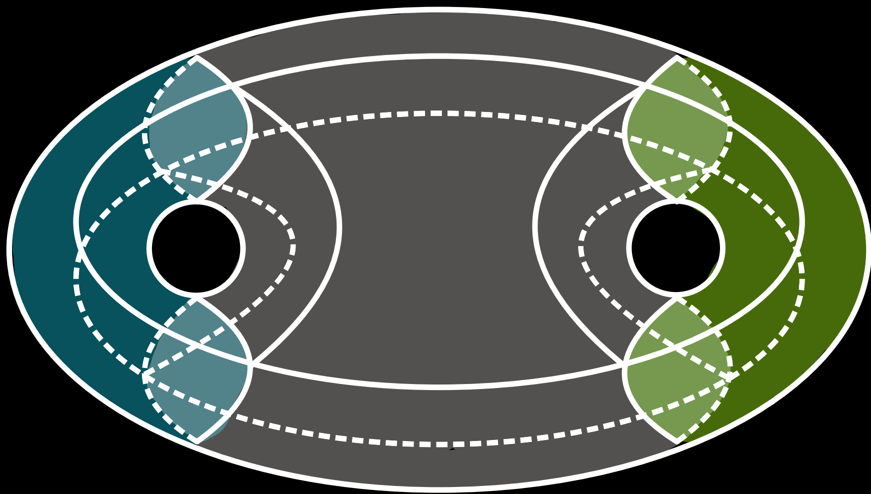
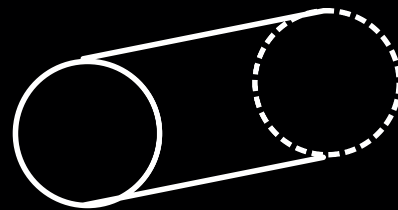
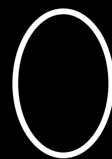
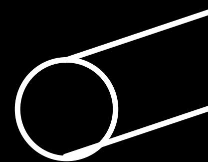

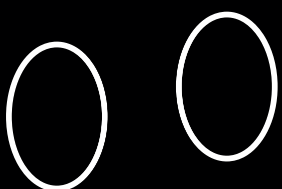
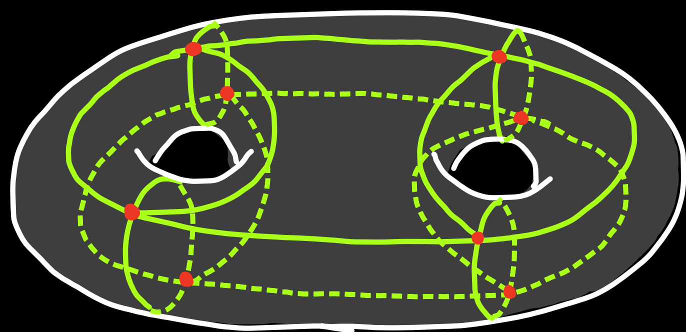
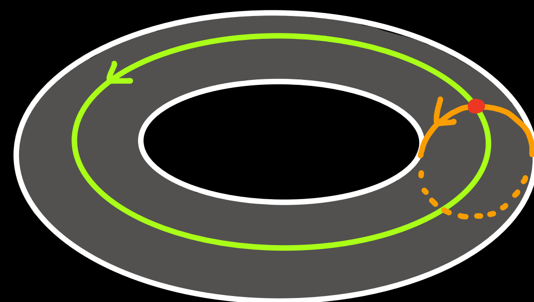
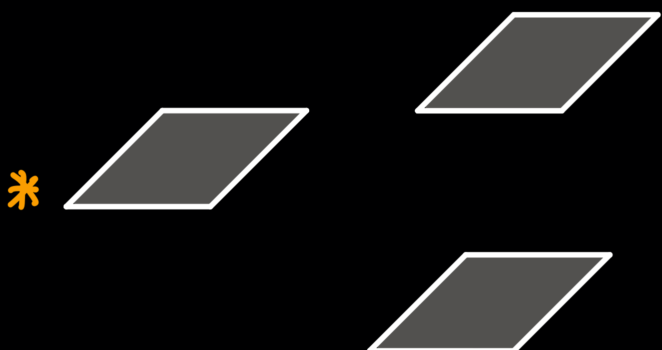
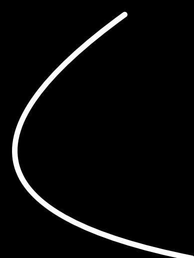
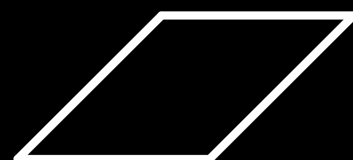

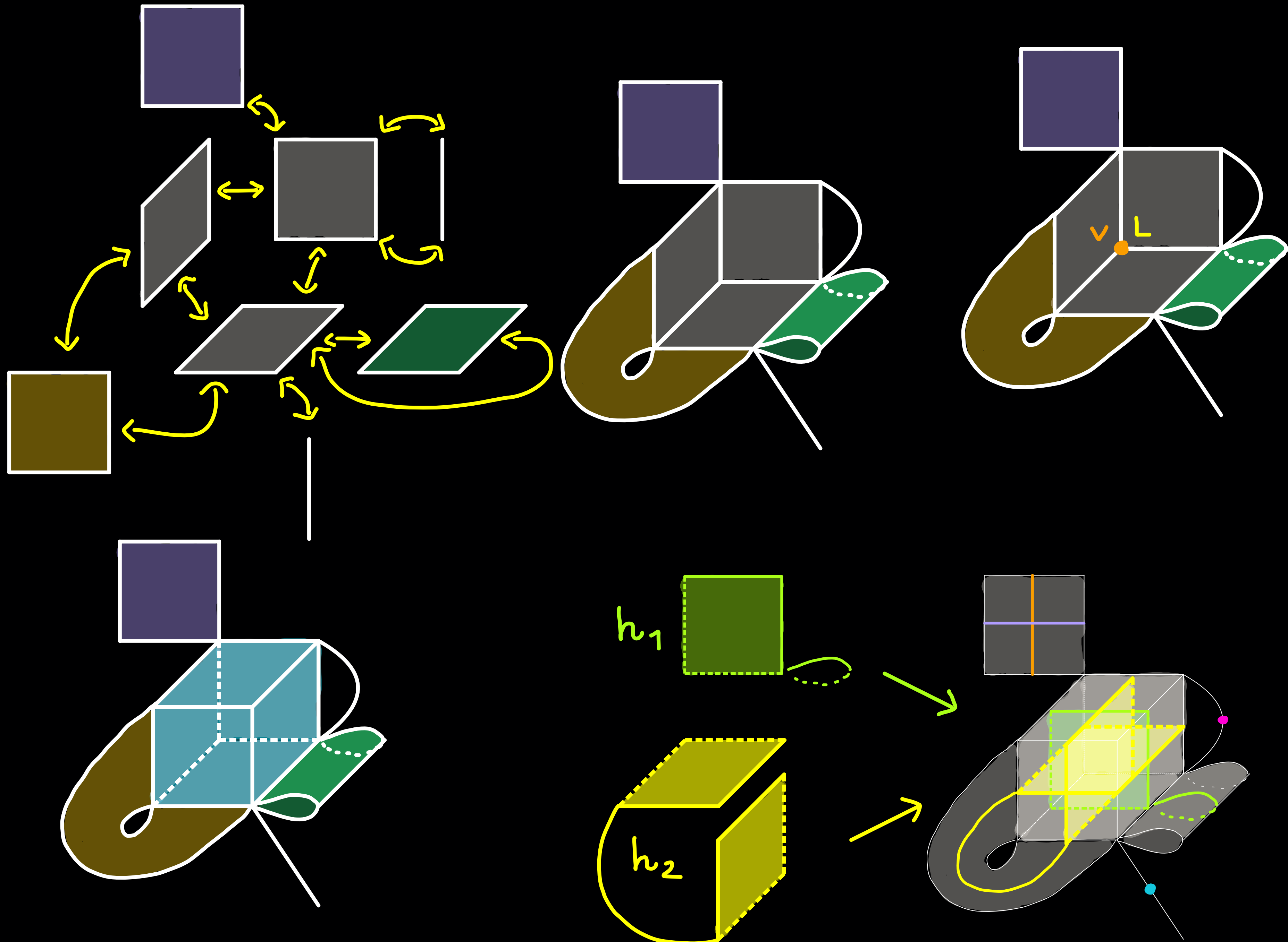
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