Approximating Hyperbolic Lattices by Cubulations

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Joint work with Nic Brody

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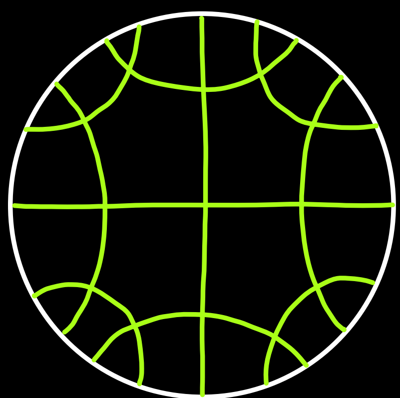
SETTING

M = closed hyperbolic n-manifold, $\Gamma = \pi_1(M)$, either
- $n \leq 3$, or
- M arithmetic of simplest type (= ∞-many tot. geodesic, codim-1 closed immersed hypersfcs Bader-Fisher-Miller-Stover'21)

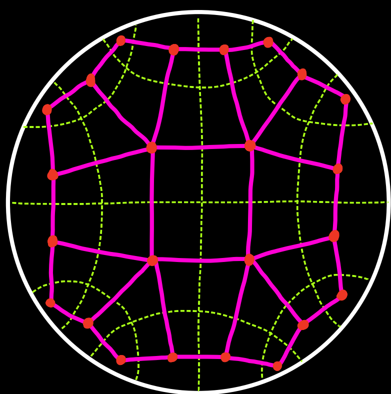
TWO TYPES OF GEOMETRIC ACTIONS
(isometric + proper + cocompact)

$\Gamma \curvearrowright \mathbb{H}^n$



Rigid if $n \geq 3$

$\Gamma \curvearrowright \mathcal{X}$ CAT(0) cube complex



Lots of them!

GOAL: Compare these two actions via Length Functions!

Length Functions

$\Gamma \curvearrowright X$ isometric action $\rightsquigarrow$ $\ell_X : \Gamma \to \mathbb{R}$

$$g \longmapsto \lim_{k\to\infty} \frac{1}{k} d_X(o, g^k \cdot o)$$

Example: (M, g) closed negatively curved manifold

$\rightsquigarrow \Gamma = \pi_1(M) \curvearrowright (\tilde{M}, \tilde{g})$ & $\ell_g(g) =$ g-length of unique geodesic in conjugacy class of g

Marked Length Spectrum Rigidity (Otal, Croke '90)

g, g_* negatively curved metrics on closed surface M

$\ell_g(g) = \ell_{g_*}(g) \;\; \forall g \;\; \longleftrightarrow \;\; g, g_*$ isometric (& isometry isotopic to identity)

Coarse MLS Rigidity (Furman '00)

Γ hyperbolic $\curvearrowright X, X_*$ geometric actions, X, X_* geodesic

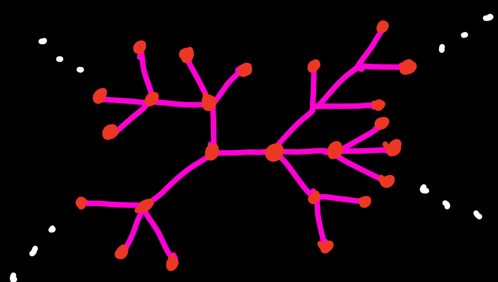
$\ell_X(g) = \ell_{X_*}(g) \;\; \forall g \;\; \longleftrightarrow \;\; \exists$ Γ-equivariant $\underbrace{\text{rough}}_{\text{up to bounded additive error}}$ isometry $X \longrightarrow X_*$

CAT(0) Cube Complexes

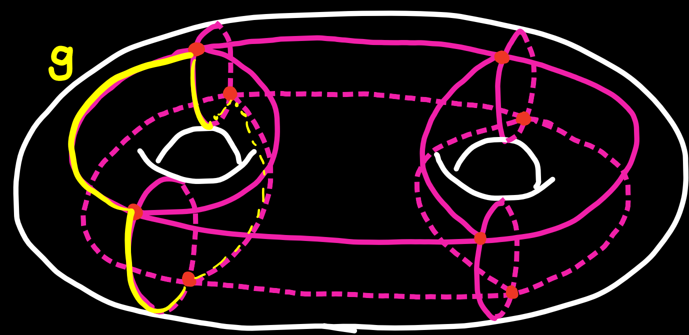
- Contractible complexes built by 1-Length Euclidean cubes isometrically along (sub)faces + Non-positive Curvature
- Geometry: Graph metric on 1-skeleton

Examples

1) Simplicial trees



2) $\bar{X}$ =



Locally CAT(0)

$\therefore$ X = universal cover is CAT(0) + $\Gamma = \pi_1(\bar{X}) \curvearrowright X$

$\ell_X(g)$ "=" minimal length of combinatorial geodesic in $\bar{X}$

Marked Length Spectrum Rigidity (Beyrer-Fioravanti: '21)

Γ hyperbolic $\curvearrowright X, X_*$ geometric actions + X, X_* "irreducible" CAT(0) cube cpxs

$$\ell_X(g) = \ell_{X_*}(g) \ \forall g \iff \exists\ \Gamma\text{-equivariant isometry } X \xrightarrow{\sim} X_*$$

Main Result

Thm (Brody-R. '24)
$\Gamma \curvearrowright \mathbb{H}^n$ torsion-free uniform lattice, s.t either:

- $n \leq 3$, or
- Γ arithmetic of simplest type

$\Rightarrow \exists$ sequence $\Gamma \curvearrowright \mathcal{X}_m$ of cubulations s.t.

$$\lambda_{1,m} \leq \frac{\ell_{\mathcal{X}_m}[g]}{\ell_{\mathbb{H}^n}[g]} \leq \lambda_{2,m} \quad \forall g \in \Gamma \quad \& \quad \lambda_{2,m}/\lambda_{1,m} \xrightarrow{m} 1$$

For m large $\mathcal{X}_m$ more Γ-equiv. "homothetic" to $\mathbb{H}^n$

Rmk: Sequence $\mathcal{X}_m$ is infinite! $\underset{\substack{\uparrow \\ \text{arithmetic}}}{\ell_{\mathcal{X}_m}}$ and $\underset{\substack{\uparrow \\ \text{non-arithmetic}}}{\ell_{\mathbb{H}^n}}$ never homothetic

Applications

$\Gamma \curvearrowright \mathbb{H}^n$ as is Setting

Cubulation of Random Quotients (+ Futer-Wise '21)

$n \leq 3 \Rightarrow$ Random quotients of Γ at density $< 1/41$ w.r.t. $\Gamma \curvearrowright \mathbb{H}^n$ are hyperbolic & cubulable

No Growth-Gap of Subgroups (+ Li-Wise '20)

$\exists$ seq $H_k < \Gamma$ quasiconvex subgroups of infinite index s.t.

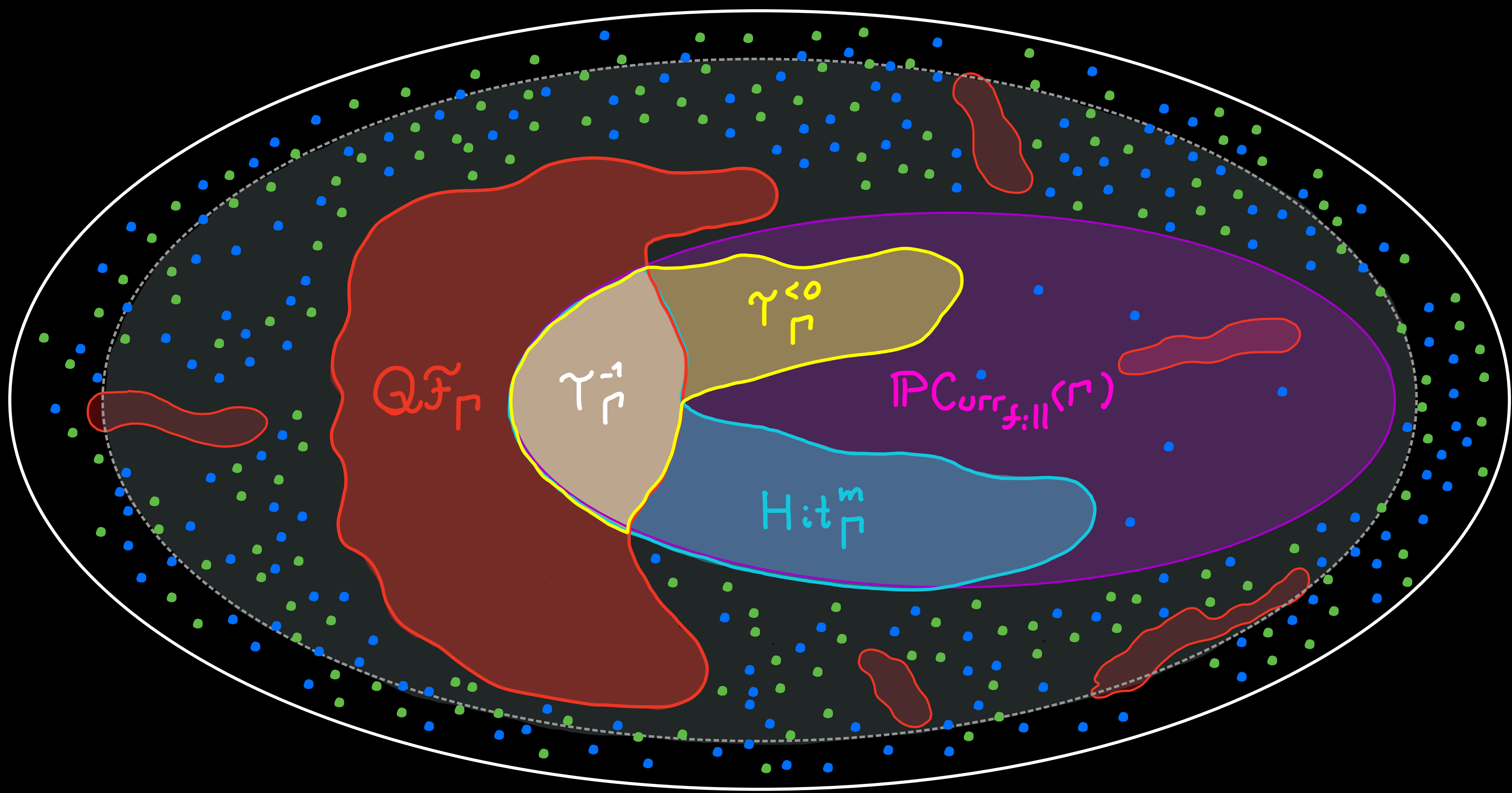
$$v(H_k) := \lim_{T \to \infty} \frac{\log \#\{h \in H_k \mid d_{\mathbb{H}^n}(o, hx) \leq T\}}{T} \xrightarrow{k \to \infty} n-1$$

Additional Approximation Results

Can also approximate by cubulations length functions for

- Negatively curved metrics on surfaces
- QuasiFuchsian reps
- Hitchin reps / maximal reps

EXAMPLE: $\mathfrak{D}_\Gamma$ for $\Gamma = \pi_1(\ \ )$



- Cayley graphs (dense)
- Green metrics (dense, Cantrell-Martínez-Granado-R. '24)
- $\mathbb{P}\mathrm{Curr}_{fill}(\Gamma)$ ($= \overline{\text{cubulations with cyclic wall stabilizers}}$)
- $\overline{\text{cubulations}}$ (contains $\mathcal{QF}_\Gamma$, Brody-R. '24)
- Anosov reps

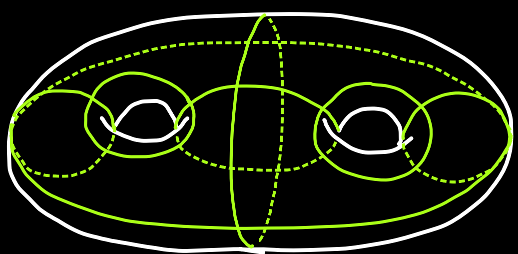
TOOLS

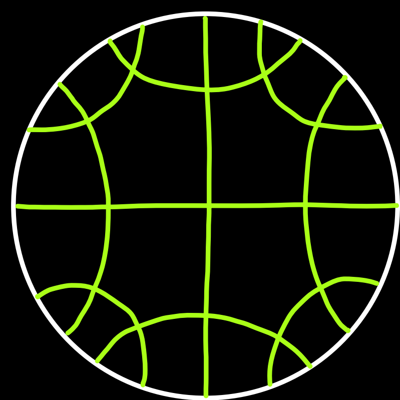
DIM = 2 ($\Gamma \curvearrowright \mathbb{H}^2$): GEODESIC CURRENTS

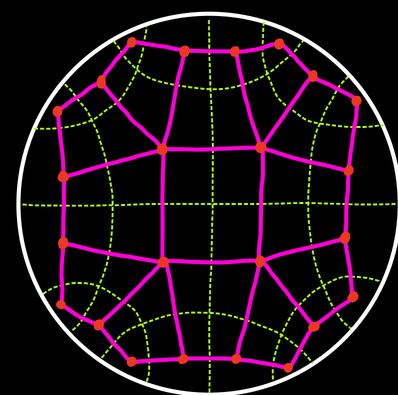
$\mathcal{G} = \{\text{geodesics in } \mathbb{H}^2\} \circlearrowleft \Gamma$

$\mathrm{Curr}(\Gamma) = \{\Gamma\text{-invariant Radon measures on } \mathcal{G}\}$

Multicurve γ on $\Gamma \backslash \mathbb{H}^2$ $\overset{\text{Lift}}{\rightsquigarrow}$ Discrete current η_γ $\overset{\text{Sageev's construction}}{\rightsquigarrow}$ Cubulation $\mathcal{X}_\gamma$ of Γ







1] $\exists$ continuous intersection number $i : \mathrm{Curr}(\Gamma) \times \mathrm{Curr}(\Gamma) \longrightarrow \mathbb{R}$

2] $\exists$ Liouville current $\mathcal{L}$ s.t. $i(\mathcal{L}, \eta_g) = \ell_{\mathbb{H}^2}(g)$

3] $\exists$ seq $h_m \in \Gamma$ s.t. $\eta_{h_m} \xrightarrow{*} \mathcal{L}$ (up to rescaling)

4] $m \gg 0 \Rightarrow \exists\, \Gamma \curvearrowright \mathcal{X}_m$ cubulation s.t. $i(\eta_{h_m}, \eta_g) = \ell_{\mathcal{X}_m}(g)$

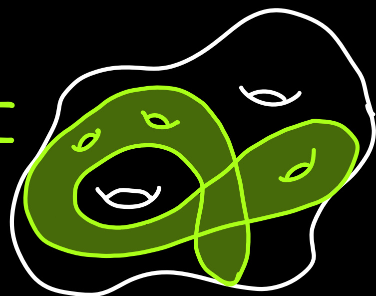
$\mathrm{Dim} = 3\ (\Gamma \curvearrowright \mathbb{H}^3)$: Co-geodesic Currents

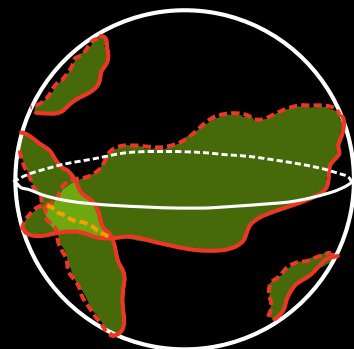
$\mathcal{QC} = \left\{ \begin{array}{l} \text{quasicircles} \\ \text{in } \mathbb{S}^2 = \partial_\infty \mathbb{H}^3 \end{array} \right\} \curvearrowleft \Gamma$

$\mathcal{QC}\mathrm{Curr}(\Gamma) = \left\{ \begin{array}{l} \Gamma\text{-invariant Radon} \\ \text{measures on } \mathcal{QC} \end{array} \right\}$

Immersed almost tot-geodesic surface $\Sigma \looparrowright \Gamma\backslash\mathbb{H}^3$ $\xrightarrow{\text{Lift}}$ Discrete co-geodesic current α_Σ $\xrightarrow{\text{Sageev's construction}}$ Cubulation $\mathcal{X}_\Sigma$ of Γ

Σ





$\tilde{\Sigma}$

1] ✓ $\exists$ "continuous" intersection number $i : \mathcal{QC}\mathrm{Curr}(\Gamma) \times \mathrm{Curr}(\Gamma) \longrightarrow \mathbb{R}$

2] ✓ $\exists$ Liouville co-geodesic current $\mathcal{L}$ s.t. $i(\mathcal{L}, \eta_g) = \ell_{\mathbb{H}^3}(g)$

3] $\exists$ seq $h_m \in \Gamma$ s.t. $\eta_{h_m} \xrightarrow{*} \mathcal{L}$ (up to rescaling)

4] ✓ $m \gg 0 \Rightarrow \exists\ \Gamma \curvearrowright \mathcal{X}_m$ cubulation s.t. $i(\eta_{h_m}, \eta_g) = \ell_{\mathcal{X}_m}(g)$

Tricky! Need minimal surface tools

Thank You!!!